ECEN620: Network Theory
Broadband Circuit Design
Fall 2014

Lecture 6: PLL Transient Behavior

Sam Palermo
Analog & Mixed-Signal Center
Texas A&M University
Announcements, Agenda, & References

- HW1 is due today by 5PM
- PLL Tracking Response
- Phase Detector Models
- PLL Hold Range
- PLL Acquisition

- Chapter 4 of *Phase-Locked Loops for Wireless Communications*, D. Stephens, Kluwer, 2002.
Linear PLL Model

• If the phase input amplitude is small, then the linear model can be used to predict the transient response

\[ E(s) = \frac{\Phi_e(s)}{\Phi_{ref}(s)} = \frac{1}{1 + \frac{G(s)}{N}} = \frac{s}{s + \frac{K_{PD}K_{VCO}F(s)}{N}} \]

• Ideally, we want this to be zero
• Phase error generally increases with frequency due to this high-pass response
First-Order PLL Tracking Response

\[ F(s) = K_1, \quad E(s) = \frac{s}{s + \frac{K_{PD}K_{VCO}K_1}{N}}, \quad K_{DC} = \omega_{3dB} = \frac{K_{PD}K_{VCO}K_1}{N} \]

- Phase Step Response

Using the Final Value Theorem:

\[
\lim_{s \to 0} \left( \frac{\Delta \Phi}{s} \right)(sE(s)) = \lim_{s \to 0} \frac{\Delta \Phi s^2}{s(s + K_{DC})} = 0
\]

Phase error should be zero with a phase step

Transient Response:

\[
\mathcal{L}^{-1}\left\{ \left( \frac{\Delta \Phi}{s} \right) \left( \frac{s}{s + K_{DC}} \right) \right\} = \Delta \Phi e^{-K_{DC}t}
\]

Transient Response is an exponentially decaying step
First-Order PLL Tracking Response

\[ F(s) = K_1, \quad E(s) = \frac{s}{s + \frac{K_{PD}K_{VCO}K_1}{N}} = \frac{s}{s + \omega_{3dB}}, \quad K_{DC} = \omega_{3dB} = \frac{K_{PD}K_{VCO}K_1}{N} \]

- Frequency Offset (Step) Response

Using the Final Value Theorem:

\[ \lim_{s \to 0} \left( \frac{\Delta \omega}{s^2} \right)(sE(s)) = \lim_{s \to 0} \frac{\Delta \omega s^2}{s^2(s + K_{DC})} = \frac{\Delta \omega}{K_{DC}} \]

The phase error is inversely proportional to the loop gain with a frequency offset

Transient Response:

\[ \mathcal{L}^{-1} \left\{ \left( \frac{\Delta \omega}{s^2} \right) \left( \frac{s}{s + K_{DC}} \right) \right\} = \frac{\Delta \omega}{K_{DC}} \left( 1 - e^{-K_{DC}t} \right) \]

Transient Response is an exponentially rising step
First-Order PLL Tracking Response

\[ F(s) = K_1, \quad E(s) = \frac{s}{s + \frac{K_{PD}K_{VCO}K_1}{N}} = \frac{s}{s + \omega_{3db}}, \quad K_{DC} = \omega_{3db} = \frac{K_{PD}K_{VCO}K_1}{N} \]

- Frequency Ramp Response

Assume that the input frequency is changing linearly with time at a rate of \( \Lambda \) (rad/sec\(^2\))

\[ \phi_{ref}(t) = \frac{\Lambda t^2}{2} \]

Using the Final Value Theorem:

\[ \lim_{s \to 0} \left( \frac{\Lambda}{s^3} \right) (sE(s)) = \lim_{s \to 0} \frac{\Lambda s^2}{s^3 (s + K_{DC})} \Rightarrow \infty \]

The phase error will grow to infinity if \( K_{DC} \) is finite

Transient Response:

\[ \mathcal{L}^{-1} \left\{ \left( \frac{\Lambda}{s^3} \right) \left( \frac{s}{s + K_{DC}} \right) \right\} = \frac{\Lambda}{K_{DC}^2} \left( K_{DC}t + e^{-K_{DC}t} - 1 \right) \]
Second-Order Type-1 PLL Tracking Response

\[ F(s) = \frac{1 + s \tau_2}{1 + s(\tau_1 + \tau_2)} \]
\[ E(s) = \frac{s + \frac{N \omega_n^2}{K_{PD} K_{VCO}}}{s^2 + 2 \zeta \omega_n s + \omega_n^2} = \frac{s + \frac{1}{\tau_1 + \tau_2}}{s^2 + \left(\frac{1 + K_{DC} \tau_2}{\tau_1 + \tau_2}\right) s + \frac{K_{DC}}{\tau_1 + \tau_2}}, \quad K_{DC} = \frac{K_{PD} K_{VCO}}{N} \]

- Phase Step Response

Using the Final Value Theorem:
\[
\lim_{s \to 0} \left( \frac{\Delta \Phi}{s} \right) (sE(s)) = \lim_{s \to 0} \frac{\Delta \Phi s^2 \left( s + \frac{1}{\tau_1 + \tau_2} \right)}{s \left( s^2 + \left(\frac{1 + K_{DC} \tau_2}{\tau_1 + \tau_2}\right) s + \frac{K_{DC}}{\tau_1 + \tau_2} \right)} = 0
\]

Phase error should be zero with a phase step

Transient Response:
\[
\mathcal{L}^{-1} \left\{ \frac{\Delta \Phi}{s} \right\} = \mathcal{L}^{-1} \left\{ \frac{s + \frac{1}{\tau_1 + \tau_2}}{s^2 + \left(\frac{1 + K_{DC} \tau_2}{\tau_1 + \tau_2}\right) s + \frac{K_{DC}}{\tau_1 + \tau_2}} \right\}
\]

Try to compute this yourself
Second-Order Type-1 PLL Tracking Response

\[ F(s) = \frac{1 + s\tau_2}{1 + s(\tau_1 + \tau_2)} \]
\[ E(s) = \frac{s + \frac{N\omega_n^2}{K_{PD}K_{VCO}}}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s + \frac{1}{\tau_1 + \tau_2}}{s^2 + \left(\frac{1 + K_{DC}\tau_2}{\tau_1 + \tau_2}\right)s + \frac{K_{DC}}{\tau_1 + \tau_2}} \], \quad K_{DC} = \frac{K_{PD}K_{VCO}}{N}

- Frequency Offset (Step) Response

Using the Final Value Theorem:
\[
\lim_{s \to 0} \left(\frac{\Delta \omega}{s^2}\right)(sE(s)) = \lim_{s \to 0} \frac{\Delta \omega s^2}{s^2 + \left(\frac{1 + K_{DC}\tau_2}{\tau_1 + \tau_2}\right)s + \frac{K_{DC}}{\tau_1 + \tau_2}} = \frac{\Delta \omega}{K_{DC}}
\]

The phase error is inversely proportional to the loop gain with a frequency offset

Transient Response:
\[
\mathcal{L}^{-1}\left\{ \left(\frac{\Delta \omega}{s^2}\right)s + \frac{1}{\tau_1 + \tau_2} \right\}
\]

Try to compute this yourself
Second-Order Type-1 PLL Tracking Response

\[ F(s) = \frac{1 + s \tau_2}{1 + s(\tau_1 + \tau_2)} \]

\[ E(s) = \frac{s + \frac{N \omega_n^2}{K_{PD} K_{VCO}}}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{s + \frac{1}{\tau_1 + \tau_2}}{s^2 + \left(\frac{1 + K_{DC} \tau_2}{\tau_1 + \tau_2}\right)s + \frac{K_{DC}}{\tau_1 + \tau_2}} \]

\[ K_{DC} = \frac{K_{PD} K_{VCO}}{N} \]

- Frequency Ramp Response

Using the Final Value Theorem:

\[ \lim_{s \to 0} \left( \frac{\Lambda}{s^3} \right) (sE(s)) = \lim_{s \to 0} \frac{\Lambda s^2 \left( s + \frac{1}{\tau_1 + \tau_2} \right)}{s^3 \left( s^2 + \left(\frac{1 + K_{DC} \tau_2}{\tau_1 + \tau_2}\right)s + \frac{K_{DC}}{\tau_1 + \tau_2} \right)} \]

The phase error will grow to infinity if \( K_{DC} \) is finite

Transient Response:

\[ \mathcal{L}^{-1} \left\{ \frac{\Lambda}{s^3} \right\} \left\{ \frac{s + \frac{1}{\tau_1 + \tau_2}}{s^2 + \left(\frac{1 + K_{DC} \tau_2}{\tau_1 + \tau_2}\right)s + \frac{K_{DC}}{\tau_1 + \tau_2}} \right\} \]

Try to compute this yourself
Second-Order Type-2 PLL Tracking Response

\[ F(s) = \frac{R\left(\frac{s + \frac{1}{RC}}{s}\right)}{s} \quad \text{and} \quad E(s) = \frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s^2}{s^2 + Ks + \frac{K}{RC}}, \quad K = \frac{K_{PD}K_{VCO}R}{N} \]

- **Phase Step Response**

Using the Final Value Theorem:

\[ \lim_{s \to 0} \left( \frac{\Delta \Phi}{s} \right)(sE(s)) = \lim_{s \to 0} \frac{\Delta \Phi s^3}{s^2 + Ks + \frac{K}{RC}} = 0 \]

Phase error should be zero with a phase step

\[
\mathcal{L}^{-1}\left\{ \frac{\Delta \Phi}{s} \left( \frac{s^2}{s^2 + Ks + \frac{K}{RC}} \right) \right\}
\]

Transient Response
Second-Order Type-2 PLL
Phase Step Response

Transient Response: \( \mathcal{L}^{-1} \left\{ \frac{\Delta \Phi}{s} \left( \frac{s^2}{s^2 + Ks + \frac{K}{RC}} \right) \right\} \)

<table>
<thead>
<tr>
<th>( \zeta )</th>
<th>( \Delta \theta )</th>
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<tbody>
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<td>( \zeta &lt; 1 )</td>
<td>( \Delta \theta \left( \cos \sqrt{1 - \zeta^2} \omega_n t</td>
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</table><p>ight) ) |
| ( \zeta &gt; 1 ) | ( \Delta \theta \left( \cosh \sqrt{\zeta^2 - 1} \omega_n tight) ) |
| ( \zeta = 1 ) | ( \Delta \theta (1 - \omega_n t) e^{-\omega_n t} ) |</p>

\[ \zeta = \frac{\omega_n}{2} \cdot \frac{1}{\sqrt{KRC}} \]
Second-Order Type-2 PLL Tracking Response

\[ F(s) = \frac{R}{s} \left( s + \frac{1}{RC} \right) \]

\[ E(s) = \frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s^2}{s^2 + Ks + \frac{K}{RC}} \]

\[ K = \frac{K_{PD}K_{VCO}R}{N} \]

- Frequency Offset (Step) Response

Using the Final Value Theorem:

\[ \lim_{s \to 0} \left( \frac{\Delta\omega}{s^2} \right) (sE(s)) = \lim_{s \to 0} \frac{\Delta\omega s^3}{s^2 + Ks + \frac{K}{RC}} = 0 \]

The phase error goes to zero with a Type-2 PLL

Transient Response:

\[ \mathcal{L}^{-1} \left\{ \left( \frac{\Delta\omega}{s^2} \right) \left( \frac{s^2}{s^2 + Ks + \frac{K}{RC}} \right) \right\} \]
Second-Order Type-2 PLL
Frequency Step Response

Transient Response: \( \mathcal{L}^{-1} \left\{ \frac{\Delta \omega}{s^2} \left( \frac{s^2}{s^2 + Ks + \frac{K}{RC}} \right) \right\} \)

\[
\zeta = \frac{\omega_n}{2RC} = \frac{1}{2} \sqrt{\frac{K}{RC}}
\]

<table>
<thead>
<tr>
<th>Frequency Step, ( \Delta \omega ) (rad/sec)</th>
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<td>( \zeta &lt; 1 )</td>
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<tr>
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<tr>
<td>( \zeta &gt; 1 )</td>
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Second-Order Type-2 PLL Tracking Response

\[ F(s) = \frac{R\left(s + \frac{1}{RC}\right)}{s}, \quad E(s) = \frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s^2}{s^2 + Ks + \frac{K}{RC}}, \quad K = \frac{K_{PD}K_{VCO}R}{N} \]

- Frequency Ramp Response

Using the Final Value Theorem: \[ \lim_{s \to 0} \left( \frac{\Lambda}{s^3} \right)(sE(s)) = \lim_{s \to 0} \frac{\Lambda s^3}{s^2 + Ks + \frac{K}{RC}} = \frac{\Lambda}{\omega_n^2} \]

A second-order type-2 PLL can track a frequency ramp with a dynamic phase lag

Transient Response: \[ \mathcal{L}^{-1}\left\{ \left( \frac{\Lambda}{s^3} \right) \left( \frac{s^2}{s^2 + Ks + \frac{K}{RC}} \right) \right\} \]
Second-Order Type-2 PLL
Frequency Ramp Response

Transient Response: $\mathcal{L}^{-1}\left\{\left(\frac{\Lambda}{s^3}\right)\left(\begin{array}{c}
\frac{s^2}{s^2 + Ks + \frac{K}{RC}}
\end{array}\right)\right\}$

| Frequency Ramp, $\Lambda$ (rad/sec²) |
|-----------------|-----------------|
| $\zeta < 1$     | $\frac{\Lambda}{\omega_n^2} - \frac{\Lambda}{\omega_n^2} \left(\cos \sqrt{1 - \zeta^2} \omega_n t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \sqrt{1 - \zeta^2} \omega_n t\right)e^{-\zeta \omega_n t}$ |
| $\zeta = 1$     | $\frac{\Lambda}{\omega_n^2} - \frac{\Lambda}{\omega_n^2} (1 + \omega_n t)e^{-\omega_n t}$ |
| $\zeta > 1$     | $\frac{\Lambda}{\omega_n^2} - \frac{\Lambda}{\omega_n^2} \left(\cosh \sqrt{\zeta^2 - 1} \omega_n t + \frac{\zeta}{\sqrt{\zeta^2 - 1}} \sinh \sqrt{\zeta^2 - 1} \omega_n t\right)e^{-\zeta \omega_n t}$ |

$\zeta = \frac{\omega_n}{2} \frac{RC}{2} = \frac{1}{2} \sqrt{KRC}$
Ideal Phase Detector

- An ideal phase detector has the same gain (slope) over a $\pm 2\pi$ range
- This allows the linear PLL model to be used for all phase relationships
Real Phase Detectors

- Many phase detectors are nonlinear and do not display the same gain for a given phase relationship.

- This implies that the PLL cannot be described by the linear model for large input phase deviations.

Figure 5.13  Phase detector s-curves.
PLL Frequency Step Response: Linear vs Behavioral Model

- Due to non-linearities in loop components (primarily the PD), a real PLL’s response can vary significantly from the linear model.
PLL Hold Range (Sinusoidal PD)

- A PLL **Hold Range** is the input frequency range over which the PLL can maintain static lock

\[ \phi_e = \frac{\Delta \omega}{K_{DC}} \]

w/ Linear Model the Steady - State Phase Error is

- First - Order: \( K_{DC} = \frac{K_{PD}K_{VCO}K_1}{N} \)
- Second - Order Type - 1: \( K_{DC} = \frac{K_{PD}K_{VCO}}{N} \)
- Second - Order Type - 2: \( K_{DC} = \infty \)

With a sinusoidal phase detector, the phase error is

\[ \sin\phi_e = \frac{\Delta \omega}{K_{DC}} \]

Since sine cannot exceed 1, the lock frequency is constrained to \( |\Delta \omega| \leq K_{DC} \)

**Hold Range:** \( |\Delta \omega_H| = K_{DC} \) (rad/sec)

- The hold range is finite for a type-1 PLL, and theoretically infinite for a type-2 PLL. However in practice it will be limited by another PLL block, such as the VCO tuning range.
First-Order PLL Phaselock Acquisition
(Sinusoidal PD)

Assuming a simple first-order PLL with a sinusoidal PD

\[ F(s) = K_1 = 1 \]

VCO Instantaneous Frequency: \( \omega_o + K_{VCO}v_c(t) \)

Sinusoidal Phase Detector Output: \( K_{PD} \sin(\phi_e) \)

Assume the input signal is at a frequency different from \( \omega_o \), such that the

input phase is \( \omega_{ref}t \) and \( \Delta \omega = \omega_{ref} - \omega_o \)
First-Order PLL Phaselock Acquisition (Sinusoidal PD)

The PLL output phase is

\[ \phi_{out}(t) = \omega_o t + \int_0^t K_{VCO} v_c(\tau) d\tau + \phi_{out}(0) = \omega_o t + \int_0^t K_{VCO} K_{PD} \sin(\phi_c(\tau)) d\tau + \phi_{out}(0) \]

The PLL phase error is

\[ \phi_e = \phi_{ref} - \phi_{out} = (\omega_{ref} - \omega_o) t - \int_0^t K_{VCO} K_{PD} \sin(\phi_c(\tau)) d\tau - \phi_{out}(0) \]

Differentiating this w.r.t. time yields the following nonlinear differential equation

\[ \frac{d\phi_c(t)}{dt} = \Delta \omega - K \sin(\phi_c(t)) \quad \text{where } K = K_{VCO} K_{PD} \]
If the PLL is locked,

$$\frac{d\phi_e(t)}{dt} = \Delta \omega - K \sin(\phi_e(\tau)) = 0$$

$$\sin(\phi_e) = \frac{\Delta \omega}{K}$$

Since sine cannot exceed $1$, the lock frequency is constrained to $\Delta \omega \leq K$

Hold Range: $|\Delta \omega_h| < K \text{ (rad/sec)}$
First-Order PLL Phaselock Acquisition
(Sinusoidal PD)

Normalization the first-order PLL differential equation by K

\[ \frac{\dot{\phi}_e}{K} = \frac{\Delta \omega}{K} - \sin(\phi_e) \]

In the phase-plane plot, there are 2 nulls where \( \frac{d\phi_e}{dt} = 0 \)

Negative-slope nulls are stable lock points,
while positive-slope nulls are unstable.

- Every cycle (2\( \pi \) interval) contains a stable null, thus \( \phi_e \) cannot change by more than one cycle before locking.
- There is no cycle slipping in the locking process.
- A cycle slip occurs when the phase error changes by more than 2\( \pi \) without locking.

Figure 8.1 Phase-plane plot of a first-order PLL (\( \Delta \omega/K = 0.5 \)).
First-Order PLL Phaselock Acquisition Time (Sinusoidal PD)

In order to find the phaselock acquisition time, we need to formally solve

$$\phi_e(t) = \Delta \omega t - \int_0^t K_{VCO} K_{PD} \sin(\phi_e(\tau)) d\tau - \phi_{out}(0)$$

If $\Delta \omega$ is zero and $\phi_e(0)$ is small, such that $\sin(\phi_e) \approx \phi_e$,

the approximate solution is the linear model phase step response

$$\phi_e(t) = -\phi_{out}(0) e^{-Kt}$$

However, if $\phi_e(0)$ is large, the response will deviate

from this linear approximation and can increase significantly
First-Order PLL Lock Failure (Sinusoidal PD)

- If the frequency offset exceeds the PLL hold range, the phase error will oscillate asymmetrically as the PLL undergoes cycle slips.
Second-Order Type-2 PLL Phaselock Acquisition (Sinusoidal PD)

Assuming a second-order type-2 PLL with a sinusoidal PD

\[ F(s) = \frac{\tau_2 s + 1}{\tau_1 s} = \frac{\tau_2}{\tau_1} + \frac{1}{\tau_1 s} \]

The filter response in the time-domain can be expressed as

\[ v_c(t) = \frac{\tau_2}{\tau_1} v_c(t) + \frac{1}{\tau_1} \int_0^t v_e(\tau) d\tau = \frac{\tau_2}{\tau_1} K_{PD} \sin(\phi_e(\tau)) + \frac{1}{\tau_1} \int_0^t K_{PD} \sin(\phi_e(\tau)) d\tau \]

The PLL output phase is

\[ \phi_{out}(t) = \omega_0 t + \int_0^t K_{VCO} v_c(\tau) d\tau + \phi_{out}(0) = \omega_0 t + K_{VCO} K_{PD} \left( \int_0^{\tau_2} \sin(\phi_e(\tau)) d\tau + \int_0^{\tau_1} \left( \frac{1}{\tau_1} \int_0^t \sin(\phi_e(\tau)) d\tau \right) d\tau \right) + \phi_{out}(0) \]
Second-Order Type-2 PLL Phaselock Acquisition (Sinusoidal PD)

The PLL phase error is

\[ \phi_e = \phi_{\text{ref}} - \phi_{\text{out}} = (\omega_{\text{ref}} - \omega_o) t - K_{VCO} K_{PD} \left( \int_0^t \int_0^{\tau_2} \sin(\phi_e(\tau)) d\tau d\tau + \int_0^t \left( \frac{1}{\tau_1} \int_0^t \sin(\phi_e(\tau)) d\tau \right) d\tau \right) - \phi_{\text{out}}(0) \]

Differentiating this twice w.r.t. time yields the following nonlinear differential equation

\[ \ddot{\phi}_e = -K_{VCO} K_{PD} \left( \frac{\tau_2}{\tau_1} \cos(\phi_e) \dot{\phi}_e + \frac{1}{\tau_1} \sin(\phi_e) \right) \]
For this Second-Order Type-2 PLL, the natural frequency and damping factor are

\[ \omega_n^2 = \frac{K_{PD} K_{VCO}}{\tau_1}, \quad \zeta = \frac{\tau_2^2 K_{PD} K_{VCO}}{4\tau_1} \]

Substituting this into the nonlinear differential equation yields the following

\[ \ddot{\phi}_e + 2\zeta \omega_n \cos(\phi_e) \dot{\phi}_e + \omega_n^2 \sin(\phi_e) = 0 \]

- No closed form solution exists, and numerical techniques are required to solve
Second-Order Type-2 PLL Phaselock Acquisition (Sinusoidal PD)

\[ \ddot{\phi}_e + 2\zeta\omega_n \cos(\phi_e) \dot{\phi}_e + \omega_n^2 \sin(\phi_e) = 0 \]

Acquisition with a phase error

\[ \phi_e \text{ and } \dot{\phi}_e \text{ vs time} \]

Phase Plane Plot: \( \dot{\phi}_e \text{ vs } \phi_e \)
Second-Order PLL Phase Plane Plots (Sinusoidal PD)

- An unstable singularity is called a Saddle Point

- A trajectory that terminates on a saddle point is called a “Separatrix”

- If a trajectory lies between the 2 separatrices, it will lock without cycle slipping

- If a trajectory lies outside the 2 separatrices, it will cycle slipping one or more times before locking (if at all)

\[
\ddot{\phi}_e + 2\zeta\omega_n \cos(\phi_e)\dot{\phi}_e + \omega_n^2 \sin(\phi_e) = 0
\]
Second-Order PLL Pull-Out Range and Lock Time (Sinusoidal PD)

• The **Pull-Out Range** is the maximum frequency step that can occur before the loop locks without cycle slipping

\[
\Delta \omega_{PO} \approx 1.8 \omega_n (\zeta + 1)
\]

for \( \zeta \) between 0.5 and 1.4

If a frequency step is less than the pull-out range, the PLL acquisition time can be approximated as

\[
t_{acq} = t_{phase} + t_{freq} = \frac{4}{\omega_n} + \frac{4.2(\Delta f)^2}{B_L^3}
\]

for phase error less than 10%

Here, \( B_L \) is the PLL noise bandwidth

\[
B_L = \int_0^\infty |H(f)|^2 df \quad \text{(Hz)}
\]

Assuming \( F(s) = \frac{\tau_2 s + 1}{\tau_1 s} = \frac{\tau_2}{\tau_1} + \frac{1}{\tau_1 s} \),

\[
B_L = \frac{\omega_n}{2} \left( \zeta + \frac{1}{4\zeta} \right) \quad \text{(Hz)}
\]
Second-Order PLL Locking Outside of the Pull-Out Range (Sinusoidal PD)

- Multiple cycle slips are observed before the loop locks
Next Time

- Phase Detector Circuits