Agenda

• PLL Stability
  • Phase & Gain Margin
  • Root Locus
  • Nyquist Plots
Feedback Configuration

Here $f = \text{feedback factor (B(s) in previous slides)}$

$$a(s) = \frac{V_o}{V_\varepsilon}(s) = \frac{a_0}{1 - \frac{s}{p_1}}$$

$$A_{CL}(s) = \frac{V_o}{V_i}(s) = \frac{a(s)}{1 + a(s)f} = \frac{a_0}{1 + a_0f} \frac{1}{1 - \frac{s}{(1 + a_0f)p_1}}$$

If $a_0$ is large: $A_{CL}(0) \approx \frac{1}{f}$ \quad $\omega_p \approx a_0p_1f$
**Magnitude & Phase**

\[ T(s) = a(s)f_2 \]

Note: \( a(s) \) can have higher-order poles

\[ |T(j\omega)|, \text{ dB} \]

- 20 \( \log a_0 f_2 \)
- 0
- 20 \( \log a_{180} f_2 \)

-20 dB/dec
-40 dB/dec
-60 dB/dec

\[ \omega_{180} \quad |p_3| \]

\[ |p_1| \quad |p_2| \]

\[ \angle T(j\omega) \]
Gain & Phase Margin

- $|a(j\omega)|$, dB
- $|T(j\omega)| = 0$ dB
- Gain margin
- Phase margin

[Karsilayan]
Stability Criteria

Nyquist:
\[ |T(j\omega_{180})| = a_{180}f < 1 \implies \text{Stable} \]

Gain Margin (GM):
\[ GM = 20 \log \left( \frac{1}{|T(j\omega_{180})|} \right) = -20 \log |T(j\omega_{180})| \]
\[ GM > 0 \implies \text{Stable} \]

Phase Margin (PM):
\[ PM = 180^\circ + \angle T(j\omega_0) \]
\[ PM > 0 \implies \text{Stable} \]
First-Order PLL

\[ F(s) = K_1 \]

Forward Path Gain: \( G(s) = \frac{K_{PD}K_{VCO}K_1}{s} = \frac{NK_{DC}}{s} \)

DC Loop Gain Magnitude: \( K_{DC} = \lim_{s \to 0} \left( \frac{sG(s)}{N} \right) = \frac{K_{PD}K_{VCO}K_1}{N} \)

Transfer Function: \( H(s) = \frac{K_{PD}K_{VCO}K_1}{s + \frac{K_{PD}K_{VCO}K_1}{N}} = \frac{N\omega_{3dB}}{s + \omega_{3dB}} = \frac{NK_{DC}}{s + K_{DC}} \)

Closed-Loop Bandwidth: \( \omega_{3dB} = \frac{K_{PD}K_{VCO}K_1}{N} = K_{DC} \)

Error Function: \( E(s) = \frac{s}{s + \frac{K_{PD}K_{VCO}K_1}{N}} = \frac{s}{s + \omega_{3dB}} = \frac{s}{s + K_{DC}} \)
First-Order PLL Stability

• Open-loop Bode plots are useful for checking stability via the phase margin

\[ G(s) = \frac{K_{PB}K_{VCO}K_1}{s} = \frac{NK_{DC}}{s} \]

\[ 20 \log_{10} \left| \frac{G(j\omega)}{N} \right| \text{ (dB)} \]

• A first-order PLL is inherently stable and always has 90° phase margin
Second-Order Type-1 PLL w/ Passive Lag-Lead Filter

\[ F(s) = \frac{1 + s\tau_2}{1 + s(\tau_1 + \tau_2)} \]

\[ \tau_1 = R_1C \quad \tau_2 = R_2C \]

Forward Path Gain : 
\[ G(s) = \frac{K_{PD}K_{VCO}(1 + s\tau_2)}{s(1 + s(\tau_1 + \tau_2))} = \frac{NK_{DC} \left( \frac{\tau_2}{\tau_1 + \tau_2} \right) (s + 1)}{s(1 + s(\tau_1 + \tau_2))} \]

DC Loop Gain Magnitude : 
\[ K_{DC} = \lim_{s \to 0} \left( \frac{sG(s)}{N} \right) = \frac{K_{PD}K_{VCO}}{N} \]

Transfer Function : 
\[ H(s) = \frac{K_{PD}K_{VCO}\tau_2 \left( s + \frac{1}{\tau_2} \right)}{s^2 + \left( 1 + \frac{K_{PD}K_{VCO}\tau_2}{N} \right) s + \frac{K_{PD}K_{VCO}}{N(\tau_1 + \tau_2)}} = \frac{\omega_n \left( 2\zeta - \frac{N\omega_n}{K_{PD}K_{VCO}} \right)s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

Natural Frequency : 
\[ \omega_n = \sqrt{\frac{K_{PD}K_{VCO}}{N(\tau_1 + \tau_2)}} \]

Damping Factor : 
\[ \zeta = \frac{\omega_n}{2} \left( \frac{N}{K_{PD}K_{VCO}} \right) \]

Error Function : 
\[ E(s) = \frac{s \left( s + \frac{N\omega_n^2}{K_{PD}K_{VCO}} \right)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]
Second-Order Type-1 PLL w/ Lag-Lead Filter Stability

- Assuming a decade spacing between filter pole and zero

\[ G(s) = \frac{NK_{DC} \left( \frac{s + \frac{1}{\tau_2}}{s + \frac{1}{\tau_1} + \tau_2} \right)}{s \left( s + \frac{1}{\tau_1} + \tau_2 \right)} \]

Normalizing KDC for \( \zeta = 1 \)

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<td>42.5</td>
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<tr>
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<td>77</td>
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<tr>
<td>10</td>
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- A larger \( K_{DC} \) provide a more stable system
Second-Order Type-1 PLL w/ Lag-Lead Filter
Output Response w/ Phase Step

- Note, time axis is scaled by $\sqrt{K_{DC}}$ in order to view the phase step plots on one graph
Root Locus

- A Root-Locus Plot is a plot of the closed-loop poles in the complex s-plane as the loop gain changes from zero to very large.

- Useful in visualizing system stability and sensitivity to variations in loop gain.

- For stability, all poles should lie within the left-half plane, i.e. no poles should be in the right-half plane.

- A good design ensures that the poles have sufficient margin from the imaginary axis for proper stability, damping, and acceptable gain peaking.
Second-Order Type-1 PLL w/ Passive Lag-Lead Filter Root Locus

Open - Loop: \( G(s) = \frac{K_{DC}}{N} \frac{\tau_2}{(\tau_1 + \tau_2)} \left( s + \frac{1}{\tau_2} \right) \frac{s + \frac{1}{\tau_1}}{s} \)

Closed - Loop: \( H(s) = N \frac{K_{DC}}{s^2 + \frac{1+K_{DC}\tau_2}{\tau_1 + \tau_2}s + \frac{K_{DC}}{\tau_1 + \tau_2}} \)

\( \tau_2 = 1, \tau_1 = 9 \)

- Initial pole values with zero loop gain are the open-loop poles
  \( p_1 = 0, \quad p_2 = -\frac{1}{\tau_1 + \tau_2} = -0.1 \)

- Final pole values with infinite loop gain are the open-loop zeros
  \( p_1 = -\frac{1}{\tau_2} = -1, \quad p_2 = -\infty \)
Second-Order Type-1 PLL w/ Passive Lag-Lead Filter Root Locus

Open-Loop: \( \frac{G(s)}{N} = \frac{K_{DC} \left( \frac{\tau_2}{\tau_1 + \tau_2} \right) \left( s + \frac{1}{\tau_2} \right)}{s \left( s + \frac{1}{\tau_1 + \tau_2} \right)} \)

Closed-Loop: \( H(s) = N \frac{K_{DC} \left( \frac{\tau_2}{\tau_1 + \tau_2} \right) \left( s + \frac{1}{\tau_2} \right)}{s^2 + \left( \frac{1 + K_{DC} \tau_2}{\tau_1 + \tau_2} \right)s + \frac{K_{DC}}{\tau_1 + \tau_2}} \)

\( \tau_2 = 1, \tau_1 = 9 \)

For \( \zeta \leq 1 \)

\( 0.1 \cdot K_1 \rightarrow \zeta = 0.38 \)

\( 10 \cdot K_1 \rightarrow \zeta = 3.09 \)

\( K_1 \text{ for } \zeta = 1 \)
Second-Order Type-1 PLL w/ Passive Lag-Lead Filter Closed-Loop Response

Open - Loop: \( G(s) = \frac{K_{DC}}{N} \frac{\left( \frac{\tau_2}{\tau_1 + \tau_2} \right) \left( s + \frac{1}{\tau_2} \right)}{s \left( s + \frac{1}{\tau_1 + \tau_2} \right)} \)

Closed - Loop: \( H(s) = N \frac{K_{DC}}{s^2 + \left( \frac{1 + K_{DC} \tau_2}{\tau_1 + \tau_2} \right) s + \frac{K_{DC}}{\tau_1 + \tau_2}} \)

20*\log_{10}|H(j\omega)|

Normalized Phase Step Response

- A larger \( K_{DC} \) provide a more stable system and wider loop bandwidth

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Second-Order Type-2 PLL  
 *w* Passive Series-RC Lag-Lead Filter

\[
F(s) = \frac{R\left(s + \frac{1}{RC}\right)}{s}
\]

Define a loop gain factor \[K = \frac{K_{PD}K_{VCO}R}{N}\]

Forward Path Gain: \[G(s) = \frac{K_{PD}K_{VCO}R\left(s + \frac{1}{RC}\right)}{s^2} = \frac{NK\left(s + \frac{1}{RC}\right)}{s^2}\]

Transfer Function: \[H(s) = \frac{K_{PD}K_{VCO}R\left(s + \frac{1}{RC}\right)}{s^2 + \left(\frac{K_{PD}K_{VCO}}{N}\right)s + \frac{K_{PD}K_{VCO}}{NC}} = \frac{N2\zeta\omega_n\left(s + \frac{\omega_n}{2\zeta}\right)}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{NK\left(s + \frac{1}{RC}\right)}{s^2 + KS + \frac{K}{RC}}\]

Natural Frequency: \[\omega_n = \sqrt{\frac{K_{PD}K_{VCO}}{NC}} = \sqrt{\frac{K}{RC}}\]

Damping Factor: \[\zeta = \frac{\omega_n}{2RC} = \frac{1}{2}\sqrt{KRC}\]

Error Function: \[E(s) = \frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\]
Second-Order Type-2 PLL w/ Passive Series-RC Lag-Lead Filter Root Locus

Open - Loop: \[ G(s) = \frac{K(s + \frac{1}{RC})}{N s^2} \]

Closed - Loop: \[ H(s) = \frac{NK(s + \frac{1}{RC})}{s^2 + Ks + \frac{K}{RC}} \]

- Initial pole values with zero loop gain are the open-loop poles
  \[ p_1 = 0 \quad p_2 = 0 \]

- Final pole values with infinite loop gain are the open-loop zeros
  \[ p_1 = -\frac{1}{RC} = -1 \quad p_2 = -\infty \]
Second-Order Type-2 PLL w/ Passive Series-RC Lag-Lead Filter Root Locus

Open-Loop: \[ \frac{G(s)}{N} = \frac{K \left( s + \frac{1}{RC} \right)}{s^2} \]

Closed-Loop: \[ H(s) = \frac{NK \left( s + \frac{1}{RC} \right)}{s^2 + Ks + \frac{K}{RC}} \]

K₁ for \( \zeta = 1 \)

10*K₁ \( \rightarrow \) \( \zeta = 3.09 \)

0.1*K₁ \( \rightarrow \) \( \zeta = 0.38 \)
Second-Order Type-2 PLL
w/ Passive Series-RC Lag-Lead Filter Stability

\[ G(s) = \frac{NK \left( s + \frac{1}{RC} \right)}{s^2} \]

Normalizing K for \(\zeta = 1\)

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\[ 20 \log_{10} \left( \frac{|G(j\omega)|}{N} \right) \text{ (dB)} \]

- A larger K provide a more stable system
Second-Order Type-2 PLL w/ Passive Series-RC Lag-Lead Filter Closed-Loop Response

\[ G(s) = \frac{K(s + \frac{1}{RC})}{s^2} \]

\[ H(s) = \frac{NK(s + \frac{1}{RC})}{s^2 + Ks + \frac{K}{RC}} \]

\[ 20\log_{10}|H(j\omega)| \]

- A larger \( K_{DC} \) provides a more stable system and wider loop bandwidth

Normalized Phase Step Response

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Typical Charge-Pump PLL Loop Filter

- A secondary capacitor $C_2$ is often added for additional filtering to reduce reference spurs
- This introduces an extra pole and potential stability concerns

$$F(s) = \begin{cases} \frac{R\left(s + \frac{1}{RC_1}\right)}{s} & \text{w/o } C_2 \\ \frac{\left(\frac{1}{C_2}\left(s + \frac{1}{RC_1}\right)\right)}{s\left(s + \frac{C_1 + C_2}{RC_1C_2}\right)} & \text{w/ } C_2 \end{cases}$$
Third-Order Type-2 PLL w/ Passive Series-RC Lag-Lead Filter & Additional Pole

\[
F(s) = \left( \frac{1}{C_2} \right) \left( s + \frac{1}{RC_1} \right) \frac{s + \frac{C_1 + C_2}{RC_1 C_2}}{s + \frac{C_1 + C_2}{RC_1 C_2}}
\]

Define a loop gain factor \( K = \frac{K_{PD}K_{VCO}R}{N} \)

Forward Path Gain: \( G(s) = \frac{K_{PD}K_{VCO} \left( \frac{1}{C_2} \right) \left( s + \frac{1}{RC_1} \right)}{s^2 \left( s + \frac{C_1 + C_2}{RC_1 C_2} \right)} = \frac{NK \left( s + \frac{1}{RC_1} \right)}{RC_2^2 \left( s + \frac{C_1 + C_2}{RC_1 C_2} \right)} \)

Transfer Function: \( H(s) = \frac{K_{PD}K_{VCO} \left( \frac{1}{C_2} \right) \left( s + \frac{1}{RC_1} \right)}{s^3 + \left( \frac{C_1 + C_2}{RC_1 C_2} \right) s^2 + \left( \frac{K_{PD}K_{VCO}}{NC_2} \right) s + \frac{K_{PD}K_{VCO}}{NRC_1 C_2}} = \frac{NK \left( s + \frac{1}{RC_1} \right)}{RC_2^2 \left( s + \frac{C_1 + C_2}{RC_1 C_2} \right)} + \frac{K}{RC_1} \)

Error Function: \( E(s) = \frac{s^2 \left( s + \frac{C_1 + C_2}{RC_1 C_2} \right)}{s^3 + \left( \frac{C_1 + C_2}{RC_1 C_2} \right) s^2 + \left( \frac{K_{PD}K_{VCO}}{NC_2} \right) s + \frac{K_{PD}K_{VCO}}{NRC_1 C_2}} = \frac{RC_2^2 \left( s + \frac{C_1 + C_2}{RC_1 C_2} \right)}{RC_2^2 \left( s + \frac{C_1 + C_2}{RC_1 C_2} \right) s^2 + Ks + \frac{K}{RC_1}} \)

If the third-pole is at a high frequency, can approximate as a second-order system with

Natural Frequency: \( \omega_n = \sqrt{\frac{K_{PD}K_{VCO}}{NC_1}} = \sqrt{\frac{K}{RC_1}} \)

Damping Factor: \( \zeta = \frac{\omega_n}{2RC_1} = \frac{1}{2} \sqrt{\frac{K}{RC_1}} \)
Third-Order Type-2 PLL w/ Passive Series-RC Lag-Lead Filter & Additional Pole Root Locus

Open-Loop: \[ G(s) = \frac{NK\left(s + \frac{1}{RC_1}\right)}{RC_2s^2\left(s + \frac{C_1 + C_2}{RC_1C_2}\right)} \]

Closed-Loop: \[ H(s) = \frac{\frac{RK\left(s + \frac{1}{RC_1}\right)}{RC_2s^3 + \left(\frac{C_1 + C_2}{C_1}\right)s^2 + Ks + \frac{K}{RC_1}}}{N} \]

R=1, C_1=1, C_2=0.1

- Initial pole values with zero loop gain are the open-loop poles:
  \[ p_1 = 0 \quad p_2 = 0 \quad p_3 = -\frac{C_1 + C_2}{RC_1C_2} = -11 \]

- Final pole values with infinite loop gain:
  \[ p_1 = -\frac{1}{RC} = -1 \quad p_{2,3} = -5 \pm j\infty \]
Third-Order Type-2 PLL w/ Passive Series-RC Lag-Lead Filter & Additional Pole Root Locus

Open-Loop: \[ G(s) = \frac{NK\left(s + \frac{1}{RC_1}\right)}{RC_2s^2\left(s + \frac{C_1 + C_2}{RC_1C_2}\right)} \]

Closed-Loop: \[ H(s) = \frac{NK\left(s + \frac{1}{RC_1}\right)}{RC_2s^3 + \left(\frac{C_1 + C_2}{C_1}\right)s^2 + Ks + \frac{K}{RC_1}} \]

R=1, C_1=1, C_2=0.1

K_1 for \( \zeta = 1 \):

0.1*K_1 → \( \zeta = 0.38 \)
Second-Order Type-2 PLL with Passive Series-RC Lag-Lead Filter Stability

\[ G(s) = \frac{NK\left(\frac{s + \frac{1}{RC}}{s^2}\right)}{s^2} \]

Normalizing K for \( \zeta = 1 \)

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- A larger K provides a more stable system
A larger K may not provide a more stable system

Normalizing K for $\zeta = 1$
Third-Order Type-2 PLL Closed-Loop Response

Open-Loop: \( G(s) \frac{N}{N} = \frac{NK\left(s + \frac{1}{RC_1}\right)}{RC_2s^2\left(s + \frac{C_1 + C_2}{RC_1C_2}\right)} \)

Closed-Loop: \( H(s) = \frac{NK\left(s + \frac{1}{RC_1}\right)}{RC_2s^3 + \left(\frac{C_1 + C_2}{C_1}\right)s^2 + Ks + \frac{K}{RC_1}} \)

\[20\log_{10}|H(j\omega)|\]

Normalized Phase Step Response

If K is increased too high frequency peaking and transient ringing occurs!
Instability and the Nyquist Criterion

Transfer function of a 3-pole amplifier:

\[ a(s) = \frac{a_0}{(1 - \frac{s}{p_1})(1 - \frac{s}{p_2})(1 - \frac{s}{p_3})} \]

Nyquist criterion for stability of the amplifier:

Consider a feedback amplifier with a stable \( T(s) \). If the Nyquist plot of \( T(j\omega) \) encircles the point \((-1, 0)\), the feedback amplifier is unstable.

For a PLL \( T(s) \) is the forward gain \( G(s) \) multiplied by the feedback factor \( \frac{1}{N} \):

\[ T(s) = \frac{G(s)}{N} \]
Nyquist Plot

For a PLL: \( T(s) = \frac{G(s)}{N} \)

Frequency Sweep of Loop Gain, \( T(s) \)

\( \omega \) positive

\( \omega = 0 \)

\( \omega = \omega_{180} \)

\( (-1,0) \)

\( a_{180}f_1 \)

\( a_0f_1 \)
Next Time

- PLL System Analysis
  - Noise Transfer Functions
  - Transient Response