Lecture 9: Frequency Response
Announcements

• HW3 is due Oct 19

• Reading
  • Razavi Chapters 3 & 6
Agenda

• Common-Source Amp Frequency Response
• Open-Circuit Time Constants (OC$\tau$) Bandwidth Estimation Technique
• Common-Drain Amp Frequency Response
• Common-Gate Amp Frequency Response
• Cascode Amp Frequency Response
Common-Source Amplifier: Low Frequency Response

\[
\frac{v_o}{v_i} = -\frac{g_{m1}}{g_{o1} + g_{o2}}
\]
Common-Source Amplifier: High Frequency Response

Small-Signal Model (Assuming $V_{G2}$ is AC gnd)

KCL @ Node $v_1 : (v_1 - v_i)G_{in} + v_i sC_{gs1} + (v_1 - v_o)s C_{gd1} = 0$

KCL @ Node $v_o : (v_o - v_1)s C_{gd1} + g_{m1} v_1 + v_o (g_o + s C_o) = 0$

where $g_o = g_{o1} + g_{o2}$

After some algebra, we get the exact transfer function:

$$
\frac{v_o}{v_i} = \frac{-g_{m} r_o \left( 1 - s \frac{C_{gd1}}{g_{m1}} \right)}{1 + s a + s^2 b}
$$

where

$$
a = R_{in} \left[ C_{gs1} + C_{gd1} (1 + g_{m} r_o) \right] + r_o \left( C_{gd1} + C_o \right)
$$

and

$$
b = R_{in} r_o \left( C_{gd1} C_{gs1} + C_{gs1} C_o + C_{gd1} C_o \right)
$$
Common-Source Amp Frequency Response

Exact Transfer Function: \[ V_o \left/ V_i \right. = \frac{-g_m r_o \left( 1 - s \frac{C_{gd1}}{g_m} \right)}{1 + sa + s^2 b} \]

For the common case when the two poles are real and far apart

Denominator \( D(s) = \left( 1 - \frac{s}{\omega_{p1}} \right) \left( 1 - \frac{s}{\omega_{p2}} \right) = 1 - s \left( \frac{1}{\omega_{p1}} + \frac{1}{\omega_{p2}} \right) + \frac{s^2}{\omega_{p1} \omega_{p2}} \approx 1 - \frac{s}{\omega_{p1}} + \frac{s^2}{\omega_{p1} \omega_{p2}} \)

Thus, \( \omega_{p1} = -\frac{1}{a} = -\frac{1}{R_{in} [C_{gs1} + C_{gd1} (1 + g_m r_o)] + r_o (C_{gd1} + C_o)} \)

and the transfer function can be approximated as a single pole system

\[ A(s) = \frac{V_o}{V_i} \approx \frac{-g_m r_o}{1 + s R_{in} [C_{gs1} + C_{gd1} (1 + g_m r_o)] + r_o (C_{gd1} + C_o)} \]
Open-Circuit Time Constants (OC\(\tau\))

- Open-circuit time constants technique can be used to estimate bandwidth
  - Much easier than deriving transfer function
  - Accurate for systems with one dominant pole

All - Pole Transfer Function: 
\[
\frac{v_o(s)}{v_i(s)} = \frac{a_0}{(\tau_1 s + 1)(\tau_2 s + 1)\ldots(\tau_n s + 1)}
\]

Denominator: 
\[b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + 1\]

Here \(b_n = \prod_{i=1}^{n} \tau_i\) and \(b_1 = \sum_{i=1}^{n} \tau_i\)

A Dominant - Pole System can be approximated as
\[
\frac{v_o(s)}{v_i(s)} \approx \frac{a_0}{b_1 s + 1} = \frac{a_0}{\left(\sum_{i=1}^{n} \tau_i\right) s + 1}
\]

Bandwidth \(\omega_h \approx \frac{1}{b_1} = \frac{1}{\sum_{i=1}^{n} \tau_i} = \omega_{h,\text{est}}\)
Open-Circuit Time Constants (OC\(\tau\))

- To compute time-constants
  1. Compute effective resistance \(R_{ko}\) facing each \(k\)th capacitor with all other caps open-circuited
  2. Form the product \(\tau_{ko} = R_{ko}C_k\)
  3. Sum all \(n\) “open-circuit” time constants

\[
\omega_{n,est} = \frac{1}{\sum_{k=1}^{n} R_{ko}C_k}
\]
Common-Source Amp w/ $OC_\tau$

Small-Signal Model (Assuming $V_{G2}$ is AC gnd)

- For $C_{gs1}$

$$R_{1o} = \frac{v_{1o}}{i_{1o}} = \frac{v_{1o}}{\left(\frac{v_{1o}}{R_{in}}\right)} = R_{in}$$

$$\tau_{1o} = R_{in} C_{gs1}$$
Common-Source Amp w/ OC$\tau$

**Small-Signal Model (Assuming $V_{G2}$ is AC gnd)**

- For $C_{gd1}$

1. \[ i_{2o} = g_m v_{gs1} + \frac{(v_{2o} + v_{gs1})}{r_o} \]

2. \[ v_{gs1} = -i_{2o} R_{in} \]

Plugging (2) into (1) and solving for \[ \frac{v_{2o}}{i_{2o}} \]

\[ R_{2o} = \frac{v_{2o}}{i_{2o}} = R_{in} (1 + g_m r_o) + r_o \]

\[ \tau_{2o} = \left( R_{in} (1 + g_m r_o) + r_o \right) C_{gd1} \]
Common-Source Amp w/ OC\(\tau\)

- For \(C_0\)

\[ R_{3o} = \frac{v_{3o}}{i_{3o}} = \frac{v_{3o}}{\frac{v_{3o}}{r_o}} = r_o \]

\[ \tau_{3o} = r_o C_o \]
Common-Source Amp w/ $OCT$

Small-Signal Model (Assuming $V_{G2}$ is AC gnd)

$3$ Time Constants: $\tau_{1o} = R_{in} C_{gs1}$, $\tau_{2o} = \left( R_{in} (1 + g_m r_o) + r_o \right) C_{gd1}$, $\tau_{3o} = r_o C_o$

$$b_1 = \sum_{i=1}^{n} \tau_i = R_{in} C_{gs1} + \left( R_{in} (1 + g_m r_o) + r_o \right) C_{gd1} + r_o C_o$$

$$\omega_{h,est} = \frac{1}{b_1} = \frac{1}{R_{in} C_{gs1} + \left( R_{in} (1 + g_m r_o) + r_o \right) C_{gd1} + r_o C_o}$$

Exactly the same as what we derived in Slide 6!

$$A(s) = \frac{v_o}{v_i} \approx \frac{-g_m r_o}{1 + s \left[ R_{in} [C_{gs1} + C_{gd1}(1 + g_m r_o)] + r_o (C_{gd1} + C_o) \right]}$$
Common-Source Amp w/ Large $R_{in}$

- Example: Using common-source output stage in a 2-stage OpAmp

$$A(s) = \frac{v_o}{v_i} \approx \frac{-g_{m1}r_o}{1 + sR_{in} \left[ C_{gs1} + C_{gd1} \left( 1 + g_{m1}r_o \right) \right] + r_o \left( C_{gd1} + C_o \right)}$$

with $R_{in} \gg r_o$

$$A(s) = \frac{v_o}{v_i} \approx \frac{-g_{m1}r_o}{1 + sR_{in} \left[ C_{gs1} + C_{gd1} \left( 1 + g_{m1}r_o \right) \right]}$$

$$\omega_{p1} = -\frac{1}{R_{in} \left[ C_{gs1} + C_{gd1} \left( 1 + g_{m1}r_o \right) \right]}$$

- Dominant pole is formed by input resistance times transistor $C_{gs}$ and $C_{gd}$ which has been multiplied by $1-A_{dc}$
  - $C_{gd}(1-A_{dc})$ is called the Miller capacitance
Common-Source Amp w/ Large $R_{in}$

- What about the second pole?

Exact Transfer Function: 
\[ \frac{v_o}{v_i} = \frac{-g_m r_o \left( 1 - s \frac{C_{gd1}}{g_{m1}} \right)}{1 + sa + s^2 b} \]

Denominator \( D(s) = \left( 1 - \frac{s}{\omega_{p1}} \right) \left( 1 - \frac{s}{\omega_{p2}} \right) = 1 - s \left( \frac{1}{\omega_{p1}} + \frac{1}{\omega_{p2}} \right) + \frac{s^2}{\omega_{p1} \omega_{p2}} \approx 1 - \frac{s}{\omega_{p1}} + \frac{s^2}{\omega_{p1} \omega_{p2}} \)

\[ \frac{1}{\omega_{p1} \omega_{p2}} = b = R_{in} r_o \left( C_{gd1} C_{gs1} + C_{gs1} C_o + C_{gd1} C_o \right) \]

\[ \omega_{p2} = -\frac{1}{\omega_{p1} R_{in} r_o \left( C_{gd1} C_{gs1} + C_{gs1} C_o + C_{gd1} C_o \right)} = -\frac{R_{in} \left[ C_{gs1} + C_{gd1} \left( 1 + g_{m1} r_o \right) \right]} {R_{in} r_o \left( C_{gd1} C_{gs1} + C_{gs1} C_o + C_{gd1} C_o \right)} \]

Assuming that the Miller Cap, \( C_{gd1} \left( 1 + g_{m1} r_o \right) \), dominates

\[ \omega_{p2} \approx -\frac{g_m C_{gd1}}{C_{gd1} C_{gs1} + C_{gs1} C_o + C_{gd1} C_o} \]
Common-Source Amp w/ Small $R_{in}$

- **Example:** Source-follower driving the common-source amp

\[
A(s) = \frac{v_o}{v_i} \approx \frac{-g_{m1}r_o}{1 + s \left( R_{in} \left[ C_{gs1} + C_{gd1}(1 + g_{m1}r_o) \right] + r_o \left( C_{gd1} + C_o \right) \right)}
\]

with $r_o \gg R_{in}$

\[
A(s) = \frac{v_o}{v_i} \approx \frac{-g_{m1}r_o}{1 + sr_o \left( C_{gd1} + C_o \right)}
\]

- Dominant pole is formed by output resistance times output capacitance plus transistor $C_{gd}$
Common-Source Amp w/ Small $R_{in}$

- What about the second pole?

**Exact Transfer Function:**

$$v_o = \frac{-g_m r_o \left(1 - s \frac{C_{gd1}}{g_{m1}}\right)}{1 + sa + s^2b}$$

**Denominator** $D(s) = \left(1 - \frac{s}{\omega_{p1}}\right)\left(1 - \frac{s}{\omega_{p2}}\right) = 1 - s\left(\frac{1}{\omega_{p1}} + \frac{1}{\omega_{p2}}\right) + \frac{s^2}{\omega_{p1}\omega_{p2}} \approx 1 - \frac{s}{\omega_{p1}} + \frac{s^2}{\omega_{p1}\omega_{p2}}$

$$\frac{1}{\omega_{p1}\omega_{p2}} = b = R_{in} r_o \left(C_{gd1}C_{gs1} + C_{gs1}C_o + C_{gd1}C_o\right)$$

$$\omega_{p2} = -\frac{1}{\omega_{p1}R_{in} r_o \left(C_{gd1}C_{gs1} + C_{gs1}C_o + C_{gd1}C_o\right)} = -\frac{r_o \left(C_{gd1} + C_o\right)}{R_{in} r_o \left(C_{gd1}C_{gs1} + C_{gs1}C_o + C_{gd1}C_o\right)}$$

$$\omega_{p2} = \frac{C_{gd1} + C_o}{R_{in} \left(C_{gd1}C_{gs1} + C_{gs1}C_o + C_{gd1}C_o\right)} \approx -\frac{1}{R_{in} \left(C_{gs1} + C_{gd1}\right)} \quad \text{(with large } C_o \text{)}$$
Common-Source Amp Frequency Response

\[ A_{dc} = -g_m r_o \quad \omega_z = \frac{g_{m1}}{C_{gd1}} \]

\[ \omega_{p1} = -\frac{1}{R_{in} [C_{gs1} + C_{gd1} (1 + g_{m1} r_o)]} \]

\[ \omega_{p2} \approx -\frac{g_{m1} C_{gd1}}{C_{gd1} C_{gs1} + C_{gs1} C_o + C_{gd1} C_o} \]

\[ \omega_{p1} = -\frac{1}{r_o (C_{gd1} + C_o)} \]

\[ \omega_{p2} \approx -\frac{1}{R_{in} (C_{gs1} + C_{gd1})} \]
Common-Source Amp Input Impedance

Neglecting Output Cap:

\[ Z_{in} = \frac{1}{[C_{GS} + (1 + g_m R_D)C_{GD}]s} \]

Input impedance is purely capacitive (C_{gs} + Miller C_{gd})

Considering Output Cap:

\[ \frac{V_X}{I_X} = \frac{1 + R_D(C_{GD} + C_{DB})s}{C_{GD}s(1 + g_m R_D + R_D C_{DB}s)} \]

Low frequency is capacitive, but then impedance experiences a zero followed by a second pole
Small signal analysis: Common-drain (source follower) amplifier

Small signal equivalent circuit

How this is done?

Why?

\[
\frac{v_{out}}{v_{in}} = \frac{g_{m1}}{g_{m1} + g_{mb} + g_{01} + g_{02}}
\]
Common-Drain Amplifier: High Frequency Response

- Simplifying the schematic a bit for SSA
  - Ideal current source load and neglecting transistor $r_o$ and $g_{mb}$ (i.e. $\lambda=\gamma=0$)
  - Will result in an optimistic DC gain estimate

[Razavi]
Common-Drain Amplifier: High Frequency Response

KCL @ Node $v_a : (v_a - v_i)G_S + v_a sC_{gd} + (v_a - v_o)sC_{gs} = 0$

KCL @ Node $v_o : (v_o - v_a)sC_{gs} - g_m(v_a - v_o) + v_o sC_L = 0$

After some algebra:

$$\frac{V_{out}}{V_{in}}(s) = \frac{g_m + C_{GSS}s}{R_S(C_{GS}C_L + C_{GS}C_{GD} + C_{GD}C_L)s^2 + (g_mR_S C_{GD} + C_L + C_{GS})s + g_m}$$
Common-Drain Amplifier: High Frequency Response

\[
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{g_m + C_{GS}s}{R_S(C_{GS}C_L + C_{GS}C_{GD} + C_{GD}C_L)s^2 + (g_m R_S C_{GD} + C_L + C_{GS})s + g_m}
\]

- From this simplified transfer function:

\[
A_{dc} = \frac{g_m}{g_m} = 1 \quad \text{(Optimistic)}
\]

**Exact**

\[
A_{dc} = \frac{g_m}{g_m + g_o + g_{mb}}
\]

\[
\omega_z = -\frac{g_m}{C_{gs}}
\]

2 poles, If we assume that they are spaced far apart:

\[
\omega_{p1} \approx \frac{g_m}{g_m R_S C_{GD} + C_L + C_{GS}} = \frac{1}{R_S C_{GD} + \frac{C_L + C_{GS}}{g_m}}
\]
Common-Drain Amp Input Impedance

\[ Z_{in} = \frac{1}{C_{GSS}} + \left(1 + \frac{g_m}{C_{GSS}}\right)\frac{1}{g_{mb} + C_{Ls}} \]

Low Frequency: \[ Z_{in} \approx \frac{1}{C_{GSS}} \left(1 + \frac{g_m}{g_{mb}}\right) + \frac{1}{g_{mb}} \]

Equivalent to a series capacitive term \( C_{gs} \left(\frac{g_{mb}}{g_m + g_{mb}}\right) \) and resistive term \( \frac{1}{g_{mb}} \)

High Frequency: \[ Z_{in} \approx \frac{1}{C_{GSS}} + \frac{1}{C_{Ls}} + \frac{g_m}{C_{GSS}C_{Ls}^2} \]

Series combination of \( C_{gs} \) and \( C_L \) and a negative resistance term \[-\left(\frac{g_m}{C_{gs}C_L\omega^2}\right)\]

The negative resistance term can be utilized in oscillator design
Common-Drain Amp Output Impedance

- Pole at very high frequency
- Zero at potentially low frequency if $R_S$ is large
  - Impedance can increase with frequency, i.e. display inductive behavior

\[
Z_{out} = \frac{V_X}{I_X} = \frac{R_S C_{GS}s + 1}{g_m + C_{GS}s}
\]
Common-Drain Amp Output Impedance

\[ Z_{out} = \frac{V_X}{I_X} = \frac{R_SC_{GS}s + 1}{g_m + C_{GS}s} \]

\[ R_2 = \frac{1}{g_m} \]
\[ R_1 = R_S - \frac{1}{g_m} \]

\[ L = \frac{C_{GS}}{g_m} \left( R_S - \frac{1}{g_m} \right) \approx \frac{R_SC_{GS}}{g_m} \text{ if } R_S >> \frac{1}{g_m} \]
Transient Behavior w/ Large $C_L$

- Inductive output impedance in combination with a large load capacitance can create undesired "ringing" in the transient response.
- If we have a large $R_S$ and $C_L$, then the assumption that we have one dominant pole is no longer valid.
- Both poles (potentially complex) should be considered in the analysis.

$$\frac{V_{out}(s)}{V_{in}} = \frac{g_m + C_{GS}s}{R_S(C_{GS}C_L + C_{GS}C_{GD} + C_{GD}C_L)s^2 + (g_m R_S C_{GD} + C_L + C_{GS})s + g_m}$$
Common-Gate Amp Low Frequency Response

- No $R_S$

\[
\frac{v_{out}}{v_{in}} = (g_m + g_{mb})R_D
\]

- With $R_S$

\[
\frac{v_{out}}{v_a} \text{ is given from left}
\]

How to get from $v_{in}$ to $v_a$?

Use amplifier input impedance and voltage divider

\[
R_{in} = \frac{1}{g_m + g_{mb}}
\]

\[
v_a = \frac{1}{R_S + \frac{1}{g_m + g_{mb}}}v_{in} = \frac{1}{1 + (g_m + g_{mb})R_S}v_{in}
\]

\[
\frac{v_{out}}{v_{in}} = \frac{v_a}{v_{in}} \frac{v_{out}}{v_a} = \frac{(g_m + g_{mb})R_D}{1 + (g_m + g_{mb})R_S} \approx \frac{R_D}{R_S} \text{ if } R_S \text{ is large}
\]
Common-Gate Amp Frequency Response

- No zero
- No Miller capacitor multiplication
- Low input impedance limits effectiveness as a voltage amplifier
- Useful as a current-to-voltage (transimpedance) amplifier

\[
\frac{V_{out}(s)}{V_{in}} = \frac{(g_m + g_{mb})R_D}{1 + (g_m + g_{mb})R_S} \left(\frac{1}{1 + \frac{C_S}{g_m + g_{mb} + R_S^{-1} s}}\right) \left(1 + R_D C_{DS} s\right)
\]
Cascode Amp Frequency Response

- If we associate the poles with the nodes A, X, and Y
  - Note, this is only an approximation, as it ignores interactions caused by “feedforward” caps (C_{gd}) and resistors
- 3 pole system

\[
\omega_{p,A} = \frac{1}{R_S \left[ C_{GS1} + \left(1 + \frac{g_{m1}}{g_{m2} + g_{mb2}}\right) C_{GD1} \right]}
\]

\[
\omega_{p,X} = \frac{g_{m2} + g_{mb2}}{2C_{GD1} + C_{DB1} + C_{SB2} + C_{GS2}}
\]

\[
\omega_{p,Y} = \frac{1}{R_D(C_{DB2} + C_L + C_{GD2})}
\]
Cascode Amp Output Impedance

- Simplified Model

Neglecting $C_{GD1}$ and $C_Y$

$Z_{out} = r_{o2} + Z_X + g_{m2} r_{o2} Z_X$

where $Z_X = r_{o1} \left| \frac{1}{s C_X} \right|

Output Impedance Pole $\omega_{Zout} = \frac{1}{r_{o1} C_X}$
Next Time

• Differential Pairs