Part II. Fundamentals of Circuit Analysis.

This is a design oriented engineering class; it is more relevant to understand circuit's operation and limitations that finding exact mathematical expressions or exact numerical solutions. Precise results can always be obtained through proper circuit simulations, and through a piece of Matlab/Maple code; however none of these programs can't design a single circuit for you; especially if you are dealing with analog circuits.

Circuit analysis techniques are considered as the fundamental tool for finding the main system parameters, and we should recognize that expert software simulators are extremely useful complementary tools. The system design procedure consists of the proper connection of active elements (mainly transistors and diodes), power supplies and passive elements (capacitors, inductor and resistors), and the selection of the operating point for each one of the active devices (transistors and diodes) to realize the desired electronic operation. For this reason, fundamentals of circuit analysis are revised in this section, but special attention is devoted to the physical interpretation of the results as well as the main properties of the fundamental topologies. It is important to study the circuits using the proper circuit analysis techniques and simulation tools, but for an engineer it is even more important to learn how to use those results for the design of efficient electronic systems. Before we discuss the properties of the electronic networks, it is necessary to introduce some important definitions such as signal components, system's transfer function, gain in decibels, magnitude response and phase response.

II.1. DC and AC signals.

A typical data acquisition system, such as the one shown in figure 2.1, consists of a sensor (transducer), a preamplifier, an analog filter and the signal processor. The transducer detects the physical quantities to be measured and processed (temperature, pressure, glucose, frequency, wireless signal, etc.); usually the sensor's output is a small signal (in the range of microvolts to millivolts; e.g. in the range of 10^{-6}-10^{-3} Volts) and it must be amplified to fit within the linear range of the analog-to-digital converter. Usually the strength of the signal is in the range of few hundredths of mill volts to few volts. Since the desired signal is usually accompanied for undesired information, the signal may be clean out through a frequency selection filter what removes most of the out-of-band information before it is converted into a digital format and further processed through dedicated software.

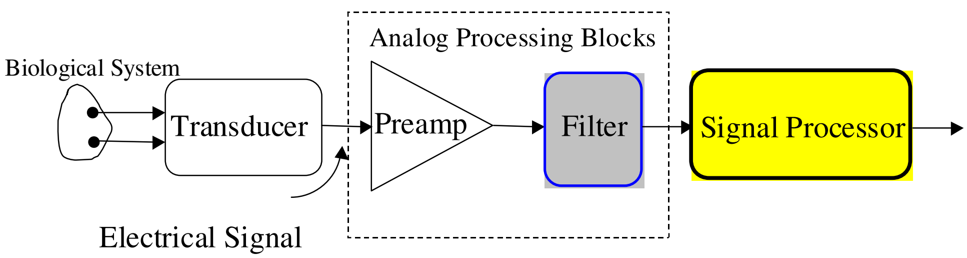


Fig. 2.1. Front-end of a typical bio-electronics system

The electrical signals resulting at the output of the transducer are usually composed by two components: the DC (Direct Current) and AC (Alternating Current) signals. As shown in Figure 2.2, the DC component is a time invariant quantity, while the AC component is a time variant quantity and ***usually this component contains the relevant information to be processed***.

We will learn in this course that the signals found in the amplifiers have both, DC and AC, components. The general expression for signals such as the one that appears at the output of the transducer in Figure 2.1 is commonly denoted as:

$$s_{AC}(t)=S_{DD}+s_{ac}(t) \qquad (2.0)$$

where S_{DD} and s_{ac} denote the DC and AC signal components, respectively. Before we discuss the manipulation techniques of these signals, let us define some of the nomenclature commonly used in electronics. The following convention is used in this text to differentiate the nature of the signal components:

$\text{CAPITAL}_{\text{CAPITAL}}$ labels are used to represent the DC component only; e.g. S_{DD}=10 V; I_X=2 Amps.
$\text{lowcase}_{\text{lowcase}}$ stands for the ac component only; e.g. $s_{ac}(t)= 2*\sin(\omega t+\theta)$; $i_{signal}(t)= 2*e^{j(\omega t+\theta)}$ Amps.

lowcase$_{CAPITAL}$ stands for the combination of both DC+ac component; e.g. $s_{AC}=S_{DD}+s_{ac}(t)$; $i_{SIGNAL}=I_X + i_{signal}$.

An example of these signals is illustrated in Figure 2.2.

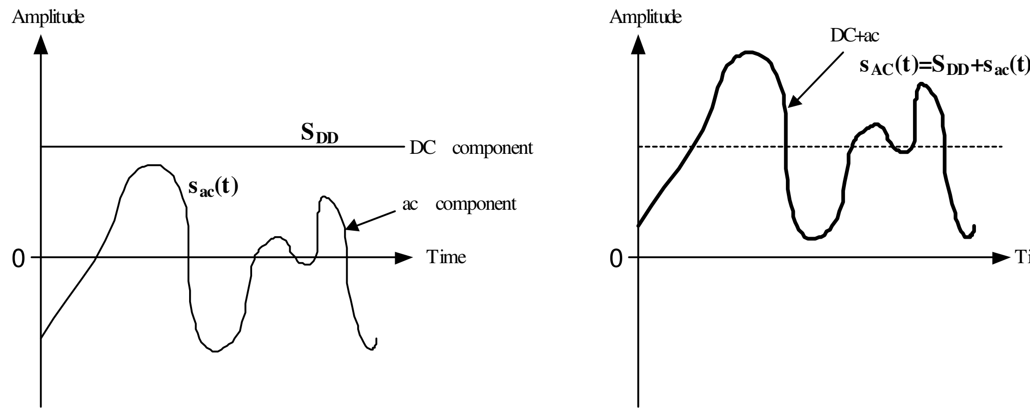


Fig. 2.2. Time domain signals: a) standalone DC and AC signals and b) combination of the signals.

Although the signals found in most of the applications are not periodic, for the analysis and design of the electronic circuits we use periodic signals (sinusoidal waveforms, pulse train, triangular waveforms, modulated waveforms, etc.) because complex waveforms present in real world applications can be approximated using periodic signals. The use of sinusoidal waveforms is especially interesting because the periodic waveforms can always be represented by a Fourier series that uses sinusoidal basis. It is well known that many practical signals are continuous and real functions f(t) (period =T and defined for all t) can be represented by the following simplified Fourier series form:

$$f(t)=\sum_{n=-\infty}^{\infty} C_n \sin(n\omega_0 t) \qquad \omega_0=\frac{2\pi}{T} \tag{2.1}$$

Where ω_0 (=2πf=2π/T) is the fundamental frequency component in radians/sec used for the series expansion. In practice, f(t) have to expressed as a linear combination of sine and cosine waveforms, but to simplify our discussion let us ignore the cosine functions.
C_n is the nth Fourier coefficient, and it is computed as follows:

$$C_n=\frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-jn\omega t}\,dt \tag{2.2}$$

Equations 2.1 and 2.2 are just a simplified form for real signal representation using the Fourier series. Examples on the use of Fourier (or Laplace) series can be found in a number of textbooks that deal with signal processing, circuit analysis and circuit realizations.

Notice in 2.1 than C_0 (n=0) represents the DC component (average) of f(t) obtained as $C_0=\frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt$. The coefficient C_1 is the fundamental (AC) component of the signal representation; for practical purposes C_1 (n=1) carriers the most relevant information to be processed since this is the fundamental frequency component of the information. The amplitude of the other terms, determined by $C_{n;\ n>1}$ are known as the high frequency harmonic components and will be rarely studied in this introductory course; that does not mean that those components are not relevant. Actually the high order terms generate undesired interferences and limit system performance.
Equations 2.1 and 2.2 are quite relevant since instead of analyzing the circuits for all possible input signals***, it is preferred to analyze them for the case of sinusoidal inputs and then infer from those results system behavior for any other kind of input signal; this is the so-called frequency domain analysis***. In the frequency domain, usually we extend the analysis of the signals from DC (ω=0) up to very high frequencies (ω even close to ∞). Although the frequency response of a system can be accurate computed and predicted, it is often not practical to find the exact

system's transfer function especially if the mathematical representation is quite complex; too much effort for very little information. A real electronic system may consist of more than 10 million transistors! In these cases it is important to have a good understanding of top circuit's behavior rather than being focused in minor details. This is an engineering course where it is more relevant to understand circuit's operation and limitations than exact mathematical derivations. Precise results can always be obtained through proper circuit simulations, but remember that the simulator can't design for you, especially analog circuits.
Few observations on the properties of logarithmic functions lead to some useful tools for plotting such complex functions. In some areas such as power electronics, the time-domain analysis is more commonly used than frequency domain analysis. Pulse and impulse system responses are employed in those cases; these topics will also be partially covered in the following subsections.

II.2 Decibels and Bode Plots.

i) Magnitude response. In electronics, the magnitude response is usually plotted in decibels (logarithmic scale); often we have to compare strong and weak signals, and it is hard to visualize their differences if linear scales are used; in these cases it is very convenient to use logarithmic scales. The magnitude of the signal or function f(x) in decibels (dB) is defined as follows:

$$\boxed{|f(x)|(dB) = 10\,log10\left(|f(x)|^2\right) = 20\,log10\left(|f(x)|\right)} \tag{2.3}$$

Where $|f(x)|$ stands for the magnitude of f(x). The expression 10*log10 ($|f|^2$) is used in communication systems where the power of the signal ($|f|^2$) is used, while the second definition 20*log10 ($|f|$) is more popular in baseband electronic applications (audio and video frequencies) where the main components are either voltage or current. In any case, the use of dB (logarithmic scale) is very convenient. Some of the properties of the logarithmic function are given in table 2.1. **Most probably you are familiar with these properties, but be sure you master these properties since will be extensively used during this course.**

Function	Properties
Log10(1)	0
Log10(10)	1
Log10(10^N)	N
Log10($1/10^N$)	-N
10*Log10(1/2) = 20*Log10($2^{-1/2}$)	**- 3.01**
10*Log10(2) = 20*Log10($2^{1/2}$)	**+ 3.01**
Log10(f*g)	Log10(f) + Log10(g)
Log10(f/g)	Log10(f) - Log10(g)
Log10(f^N)	N*Log10(f)
Log10(Af^X/Bg^X)	**Log10(Af^X) – Log10(Bg^X)** **=Log10(A)+X Log10(f)- Log10(B)-X Log10(g)**

Table 2.1 Fundamental properties of the logarithmic function

Notice that:

- 10*Log10(1) = 0 dB; it is important to recognize that 0 dB means unity in linear scale!
- The magnitude of $2^{1/2}$ (=1.414) correspond to + 3dB and $2^{-1/2}$ (=0.7071) correspond to - 3dB.
- The multiplication of functions in linear scale is converted into addition of logarithmic functions; it is a lot easier to manipulate and have good intuition when we deal with complex functions.

- The ratio of two functions is mapped into the subtraction of two logarithmic functions.
- The exponential function is mapped into scaling factors of the logarithm of the original function.

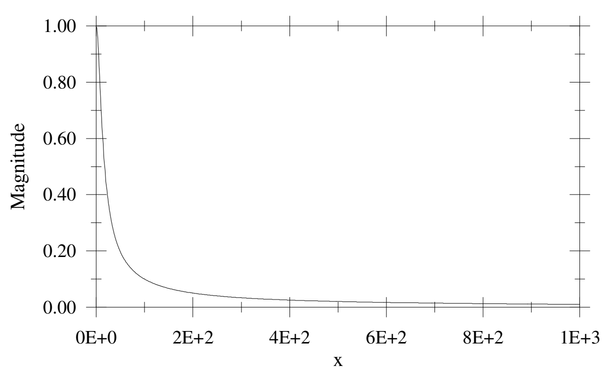
To appreciate the benefits of the logarithmic scale, let us consider the following complex function:

$$f(x) = \frac{A_1}{A_2 + jx} \tag{2.4}$$

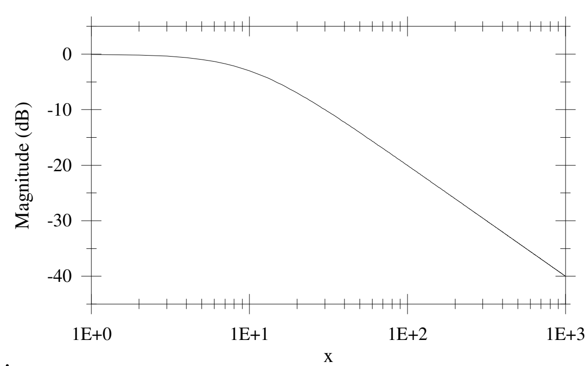
where A_1 and A_2 are real numbers, and jx is the imaginary part of the denominator; $j^2 = -1$; x is a variable that can take values in the range of $\{-\infty < x < \infty\}$. ***A_2 is defined as the pole's frequency for this transfer function; function shape is further defined by this parameter***. The squared magnitude (power) of this function can be expressed as:

$$|f(x)|^2 = f(x)f^*(x) = \frac{A_1^{\,2}}{A_2^{\,2} + x^2} \tag{2.5}$$

Where f*(x) is the complex conjugate of f(x). The magnitude response (plot of the magnitude of f(x) versus x) of this function, in linear scale, with $A_1=A_2=10$ is depicted in Figure 2.3a. Notice that it is very hard to extract information from this plot; for very small x, the value of f(x) is close to unity and decreases when x approaches 10; it is very difficult to accurately predict the value of f(x) for $x>A_1=A_2=10$; we can, of course, say that the tendency of the function goes to zero but how close is hard to measure.
The same transfer function is plotted in dB in Figure 2.3b; notice that X-axis is in logarithmic scale. For this plot, expression 2.5 is inserted into 2.3 and plotted. It is obvious that we can easily extract much more information from this plot, especially if we keep in mind some of the properties of the logarithmic function. For instance, the function is approximately flat from very small x-values up to x=5, and the -3 dB value is around x=10 (f(10)=0.707). The gain of the transfer function at x=1000 is -40 dB, which corresponds to a magnitude equal to 0.01; check this value using your calculator and equation 2.3.



(a)



(b)

Figure 2.3. Magnitude response of the function 10/(10+jx) using a) linear-linear scale and b) dB-log scale.

To ***get more insight*** into the transfer function analysis by inspection, let us further study the previous function. Taking advantage of the properties of the logarithmic function, the magnitude of the transfer function given in eqn. 2.5 can be expressed in dB as follows:

$$|f(x)|(\mathrm{dB}) = 10 * \log 10\left(A_1^{\,2}\right) - 10 * \log 10\left(A_2^{\,2} + x^2\right) \tag{2.6}$$

Notice that:

- For x << A_2 this equation can be approximated as $20*\log10(A_1)-20*\log10(A_2) = 20*\log10(A_1/A_2)$; (=0 dB for $10=A_1=A_2$).

- For $x=A_2=10$, the magnitude of f(x) is equal to
  $|f(x)|=10*\log10(A_1^2)-10*\log10(A_2^2*2)=$
  $|f(x)|=20*\log10(10)-10*(\log10(10^2)+\log10(2))$
  $|f(x)|=20*\log10(10)-20*\log10(10)-10*\log10(2)$
  $|f(x)|= -10*\log10(2) = -3.01$ dB.
- For $x >> A_2$ equation 2.6 can be approximated as $10*\log10(A_1^2)-10*\log10(x^2)$ = **20log10(10) - 20log10(x)** dB. Notice that the function consists of a constant term and a negative term that decreases proportional to the -20log10 function of x. If we use log10 scale on the x-axis, this corresponds to a straight line with an slope of –20, which fits very well with the plot shown in figure 2.3b for $x>> A_2=10$. If x=1000, then $|f(1000)| \sim 20-20*\log10(10^3)=-40$ dB which fits very well with the results shown in figure 2.3b.
- The results can be easily extrapolated to any x value.

The transfer function can be easily plotted making some additional observations. For frequencies beyond the pole's frequency (A_2 in our previous example) |f(x)| is approximately equal to $20*\log10(A_1) -20*\log10(x)$ dB; some values are given in the next table for the previous case ($A_1=A_2=10$).

x	f(x)≅20-20log10(x) (dB)
10^3	-40
10^4	-60
10^5	-80

Table 2.2 Values for the magnitude of the function 10/(10+jx) in dB.

Notice that ***beyond the frequency of the pole a variation of 1 decade in x-axis corresponds to an attenuation of –20 dB***; hence ***the slope of the function, when log10 scale is used for the X-axis is –20 dB/decade***. It can also be shown that ***a variation of 2 (1 octave) in the X-axis corresponds to an attenuation of –6 dB in |f(x)|, leading to a slope of –6dB/octave***; the scale in octaves is frequently used in music and medical equipment, and gives a little bit more resolution (variations of 2 in the x-axis) compared with the case of dB-scale. In any case, we are just exploring easy ways to plot complex functions and identifying the properties of their magnitude.
ii) The phase response of the first order complex function (solid line) is shown in figure 2.4 ($A_1=A_2=10$); the phase shift of the complex function given in equation 2.4 is computed from the following expression

$$Phase(f(x))\big|_{radians} = Phase\ (\ Numerator\ ) - Phase(Denominator) = \\ = Phase\ (\ A_2\ ) - Phase(A_2 + jx) = 0 - tan^{-1}\left(\frac{x}{A_2}\right) \qquad (2.7a)$$

The phase computed from this expression is given in radians; to obtain the phase in degrees you must multiply this value by 360/2π~57.3. The phase plot is shown in fig. 2.4; this is non-linear function. Some observations here will allow us to approximate this function using simpler but quite useful piecewise linear functions:

- The phase at x=0 is 0 degrees because the function is real and positive. Negative real functions at x=0 will start in π radians or 180 degrees.
- The phase shift of f(x) at the location of the pole $x=A_2$ is –45 degrees.
- The phase at x=∞ is -π/2 radians or –90 degrees.
- If we evaluate the derivative of the phase function at the pole's location ($x=A_2$) it becomes the largest value and corresponds to more than –45 degrees/decade.

In many practical cases, and it is the case in this course, the phase of the function is usually plotted in degrees. For converting radians to degrees the following expression is employed

$$Phase(f(x))\Big|_{deg\ reess} \cong 57.3 * Phase(f(x))\Big|_{radians} \tag{2.7b}$$

The phase response of the transfer function (in degrees) given by 2.7a can be approximated by a piece-wise linear function such that below x=0.1*A_2 the phase is approximated by 0 degrees and above x=10*A_2 it is approximated by –90 degrees. The ***phase shift around x=A_2=10 (the frequency of the pole) is forced to have a slope of –45 degrees per decade***. Both the exact phase response (solid trace) and the piece wise linear approximation (dashed trace) are shown in figure 2.4. The largest error introduced by the approximation is +/- 5.7 degrees, which is acceptable for most hand calculations.

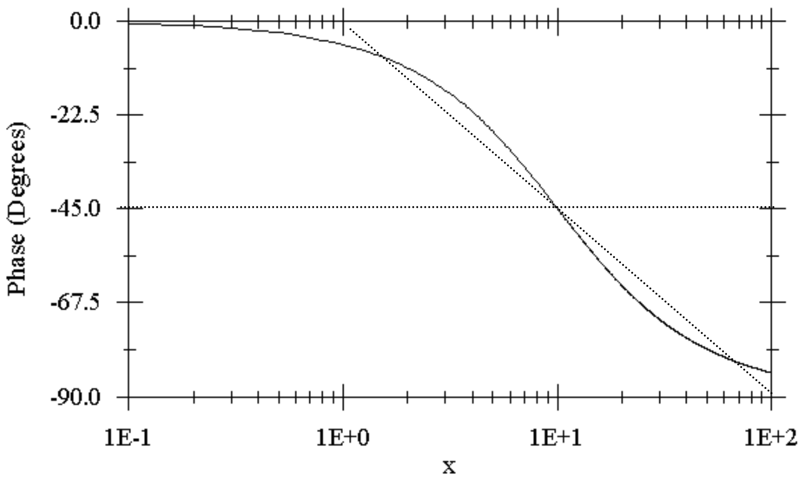


Figure 2.4. Phase response for equation 2.2 with A_1=A_2=10.

The previously discussed properties hold for the general case; let's for instance consider the following complex function:

$$f(x) = \frac{A_1}{A_2 + jA_3 x} \tag{2.8}$$

This equation can be manipulated as follows:

$$f(x) = \frac{\frac{A_1}{A_3}}{\frac{A_2}{A_3} + jx} \tag{2.8b}$$

Therefore, this equation becomes similar to eqn. 2.5. |f(x)| in dB can be expressed as

$$|f(x)|(dB) = 10 * \log 10\left(\frac{A_1^{\,2}}{A_3^{\,2}}\right) - 10 * \log 10\left(\frac{A_2^{\,2}}{A_3^{\,2}} + x^2\right) \tag{2.9}$$

The transfer function can be easily sketched if we observe that:

$$\left|f(x)\right|(dB)=\begin{cases} 10*\mathbf{log}10\left(\dfrac{A_1^{\,2}}{A_3^{\,2}}\right)-10*\mathbf{log}10\left(\dfrac{A_2^{\,2}}{A_3^{\,2}}\right)=20*\mathbf{log}10\left(A_1^{\,2}\right)-20*\mathbf{log}10\left(A_2^{\,2}\right) & \text{if } x<<\dfrac{A_2}{A_3} \\ 10*\left(\mathbf{log}10\left(A_1^{\,2}\right)-\mathbf{log}10\left(A_2^{\,2}\right)\right)-3 & \text{if } x=\dfrac{A_2}{A_3} \\ 10*\left(\mathbf{log}10\left(\dfrac{A_1^{\,2}}{A_3^{\,2}}\right)-\mathbf{log}10\left(x^2\right)\right)\cong 20*\mathbf{log}10\left(\dfrac{A_1}{A_3}\right)-20*\mathbf{log}10\left(\left|x\right|\right) & \text{if } x>>\dfrac{A_2}{A_3} \end{cases} \qquad (2.10)$$

)

To plot the transfer function is straight-forward, as shown in Figure 2.5. To have an idea of the transfer function, a piece-wise linear approximation can be used. The magnitude of the transfer function can be approximated as constant (=10*log10(A_1^2) -10*log10(A_2^2) =20*log10(A_1/A_2)) up to x=A_2/A_3; after this frequency, the magnitude of the transfer function decreases with a slope of –20 dB/decade. If the linear approximation is used, a maximum error of –3 dB at x=A_2/A_3 is introduced. For most of our practical applications we will use the piece-wise linear approximation. It should be evident that the parameter A_2/A_3 plays an important role on the shape of the transfer function; this parameter is in fact the pole's location of the function and is further discussed in the following sections.

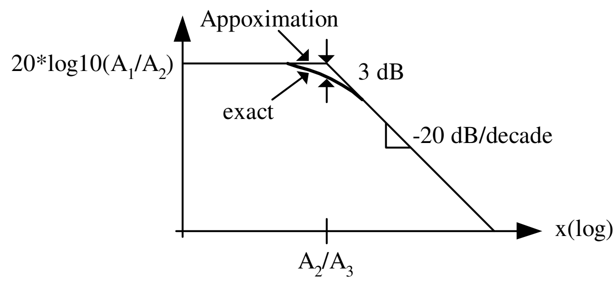


Fig. 2.5. Plot of the transfer function described by equation 2.9. Notice that a linear approximation can be used with a maximum error of –3 dB at the pole's frequency.

Since f(x) is complex, we have to consider its phase response as well. Considering f(x) given by equation 2.5, the phase response is

$$Phase(f(x)) = Phase(A_1) - tan^{-1}\left(\frac{x}{A_2/A_3}\right) = -tan^{-1}\left(\frac{A_3 x}{A_2}\right) \qquad (2.11)$$

This function is similar to the one plotted in Figure 2.4. For x=0, the phase shift presented by the circuit is 0 degrees. It is evident from equation 2.11 and this plot that the phase shift is –0.785 radians (=-45 degrees) for x= A_2/A_3; the phase shift when x → ∞ is -1.57 radians (-90 degrees). Notice that the phase shift is close to the limits of the piece wise linear approximation (~ 0º if x~ 0.1*A_2/A_3, and -90º if x~ 10*A_2/A_3). The transition from 0 to –90 degrees occurs, approximately, within 2 decades around the system's pole x=A_2/A_3. The ***slope is approximately –45 degrees/decade***; the maximum error is +/- 5.7 degrees. ***The phase shift is -45 degrees at x=A_2/A_3***. The overall phase response is similar to the one shown in Fig. 2.4 with the –45 degree frequency at x=A_2/A_3.

iii) Magnitude and phase of a general function. The complex transfer function might consist of a combination of poles and zeros. The benefits of the logarithmic scale are more evident in these cases. In the previous examples we found that a pole reduces the magnitude of the transfer function with a roll-off of –20 dB/decade beyond the pole's

location, in addition to the negative phase shift. A zero, on the other hand, increases the transfer function with a slope of +20 dB/decade and introduce positive phase shift.
Let us consider the case of a first order transfer function with a pole and a zero such as:

$$f(x)=\frac{A_1+jA_2x}{A_3+jA_4x}=\left(\frac{A_2}{A_4}\right)\left(\frac{\frac{A_1}{A_2}+jx}{\frac{A_3}{A_4}+jx}\right) \tag{2.12}$$

A_1/A_2 and A_3/A_4 are termed the zero and pole, respectively, of the complex function. The zero corresponds to the positive value of X(=jx) that results of the solution of $X_z+A_1/A_2=0$; or $X_z=-A_1/A_2$. Similarly the pole corresponds to the positive value of X_p from $X_p+A_3/A_4=0$ or $X_p=-A_3/A_4$.
The magnitude (in dB) of 2.12 can be expressed as follows:

$$|f(x)|(dB)=10*\log 10\left(\left(\frac{A_2}{A_4}\right)^2\right)+10*\log 10\left(\left(\frac{A_1}{A_2}\right)^2+x^2\right)-10*\log 10\left(\left(\frac{A_3}{A_4}\right)^2+x^2\right) \tag{2.13}$$

Similar to the case of a single pole function, the three terms of this expression can be approximated for different x ranges. Let us ***consider the case $A_1/A_2<A_3/A_4$***, in this case location of zero< location of pole. From 2.13 it follows that

$$|f(x)|(dB)=\begin{cases}10*\log 10\left(\frac{A_2}{A_4}\right)^2+10*\log 10\left(\frac{A_1}{A_2}\right)^2-10*\log 10\left(\frac{A_3}{A_4}\right)^2=10*\log 10\left(\frac{A_1}{A_3}\right)^2 & \text{if } x \ll \frac{A_1}{A_2}<\frac{A_3}{A_4}\\ 10*\log 10\left(\frac{A_1}{A_3}\right)^2+3 & \text{if } x=\frac{A_1}{A_2}\\ 10*\log 10\left(\frac{A_2}{A_4}\right)^2+10*\log 10\left(\left(\frac{A_1}{A_2}\right)^2+x^2\right)-10*\log 10\left(\frac{A_3}{A_4}\right)^2 & \text{if } \frac{A_1}{A_2}<x<\frac{A_3}{A_4}\\ 10*\log 10\left(\frac{A_2}{A_4}\right)^2+10*\log 10\left(\left(\frac{A_1}{A_2}\right)^2+x^2\right)-10*\log 10\left(\frac{A_3}{A_4}\right)^2-3 & \text{if } x=\frac{A_3}{A_4}\\ 10*\log 10\left(\frac{A_2}{A_4}\right)^2 & \text{if } x \gg \frac{A_3}{A_4}\end{cases}$$

. (2.14)

These results can be plotted by using the piece wise linear approximation as follows

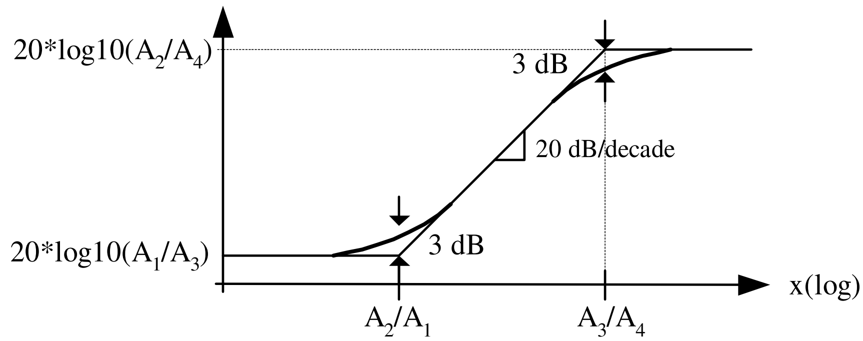


Fig. 2.6. Magnitude response of a function with a pole-zero pair; zero location < pole location.

For the plotting function phase response we can use the aforementioned properties. The phase of transfer function in radians is obtained by using the following expressions:

$$
\begin{aligned}
\text{Phase}\left(f\left(x\right)\right) &= \text{phase}\left(\frac{\dfrac{A_1}{A_2}+jx}{\dfrac{A_3}{A_4}+jx}\right) \\
&= \text{phase}\left(\frac{A_1}{A_2}+jx\right)-\text{phase}\left(\frac{A_3}{A_4}+jx\right) \qquad (2.15)\\
&= \mathbf{tan}^{-1}\left(\frac{A_2 x}{A_1}\right)-\mathbf{tan}^{-1}\left(\frac{A_4 x}{A_3}\right)
\end{aligned}
$$

Therefore, the phase of the zero and pole can be considered independently, and the overall phase is the subtraction of the phase responses; the result is shown in figure 2.7 in case the zero is located well before the pole's location $A_1/A_2 << A_3/A_4$. Since the zero is located before the pole the phase starts at 0 degrees, and at the frequency of the zero the phase shift is approximately equal to + 45 degrees. The phase goes up to +90 degrees at x=10 A_1/A_2, and remains constant until the pole start affecting the phase response. The pole introduces negative excess phase that is noticeable at x-values around the pole's location A_3/A_4. At the pole value ($x=A_3/A_4$) the phase shift due to the zero is +90 degrees while the pole contributes with -45 degrees, leading to an overall phase equal to +45 degrees (~ phase shift of +90 degrees due to the zero – 45 degrees phase shift due to the pole). At very high frequencies the phase shift is zero because the +90 degrees introduced by the zero is cancelled by the –90 degrees due to the pole. In the next subsection we consider the case of poles and zeros close to each other.

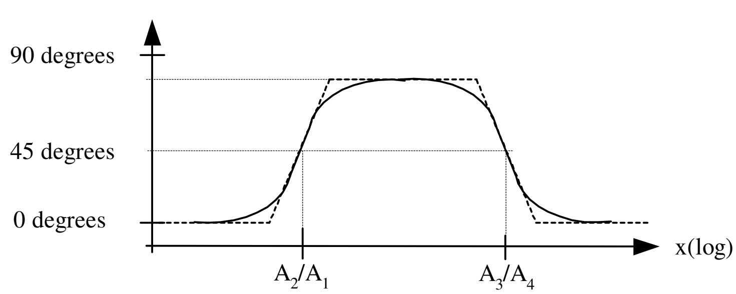


Fig. 2.7. Phase response of the first-order transfer function.

Bode plots: General case.
In most of the practical cases, the characteristic equation of the electronic system consist of several sections (multi-stage systems) connected in cascade. Although the sections may need a more complex representation, lets assume that each stage can be represent by a first order equation, hence the electronic system can be mathematically represented as:

$$f(x)=\frac{\prod_{k=1}^{N}\left(A_k+jB_kx\right)}{\prod_{i=1}^{M}\left(C_i+jD_ix\right)}=\left(\frac{\prod_{k=1}^{N}\left(B_k\right)\left(\frac{A_k}{B_k}+jx\right)}{\prod_{i=1}^{M}\left(D_i\right)\left(\frac{C_i}{D_i}+jx\right)}\right) \tag{2.16}$$

Where the symbol $\prod$ means the multiplication of the factors. This factorization is very convenient especially for low-pass transfer function, as will be evident soon. Using the properties of the logarithmic function, the magnitude (in dB) of this transfer function at DC (x=0) yields:

$$\left|f(x)\right|_{x=0}(dB)=10*\mathbf{log}10\left(\frac{\prod_{k=1}^{N}A_k^{\ 2}}{\prod_{i=1}^{M}C_i^{\ 2}}\right)$$

For any other frequency, and since we do not have information about the location of the poles and/or zeros, the complete expression must be manipulated.

$$\left|f(x)\right|(dB)=10*\mathbf{log}10\left(\frac{\prod_{k=1}^{N}B_k^{\ 2}}{\prod_{i=1}^{M}D_i^{\ 2}}\right)+10*\sum_{k=1}^{N}\mathbf{log}10\left(\left(\frac{A_k}{B_k}\right)^2+x^2\right)-10*\sum_{i=1}^{M}\mathbf{log}10\left(\left(\frac{C_i}{D_i}\right)^2+x^2\right) \tag{2.17}$$

Notice that we normalized the functions with respect to the zeros and poles to make easier the analysis of the transfer function. The role of these terms are:

1. By making x=0 in eqn. 2.17 the dc gain of the function (magnitude of the transfer function for x=0) can be obtained. This is the starting point of the overall transfer function, and depends on the coefficients A_k and C_i since other terms will cancel each other.
2. The zeros and poles must be identified. After the zero's frequency, ***each zero will increase the magnitude of the function at a rate of 20 dB/decade***. Similarly, ***each pole will decrease the magnitude of the function by -20 dB/decade after the pole's location***. Since we are using logarithmic functions and log X scale, the effects of poles and zeros can be algebraically added.
3. At a given x-value, the roll-off (slope) of the magnitude of the function can be easily obtained by the following equation:

$$slope\ f\left(x=x_0\right)=\left(\#\,zeros\,below\,x_0-\#\,poles\,below\,x_0\right)\left(20\ dB\right) \tag{2.18}$$

4. Finally, evaluate the function for x → ∞. Notice that the final value depends on several factors:

a. If the number of zeros is greater than the number of poles, the transfer function goes to infinite when $x \rightarrow \infty$; **positive infinite in dB**.
b. If the number of poles is greater than the zeros (usually this is the case in electronics), the final value is zero when $x \rightarrow \infty$; **negative infinite in dB**.
c. In case the number of poles and zeros are equal, the function final value is function of the coefficients B_k and D_i.

Similar rules can be established for the phase response. The general phase equation can be express as follows:

$$\begin{aligned}\text{Phase}\left(f\left(x\right)\right) &= \text{phase}\left(\prod_{k=1}^{N}\left(\frac{A_k}{B_k}+jx\right)\right)-\text{phase}\left(\prod_{i=1}^{M}\left(\frac{C_i}{D_i}+jx\right)\right) \\ &= \sum_{k=1}^{N}\tan^{-1}\left(\frac{B_k x}{A_k}\right)-\sum_{i=1}^{M}\tan^{-1}\left(\frac{D_i x}{C_i}\right)\end{aligned} \tag{2.19}$$

Each zero introduces +45 degrees at the zero and roughly +90 degrees 1 decade after the zero's location, while each pole introduces –45 degrees at its pole's location and –90 degrees 1 decade after. According to equation 2.19 the phase contribution of each term can be algebraically added.

Example: A numerical example helps us to have a better understanding on the use of these rules. Let us consider the following example with two poles and two zeros.

$$f(x)=\left(\frac{500+j10x}{1000+j100x}\right)\left(\frac{1000+j50x}{20000+j200x}\right) \tag{2.20}$$

Basic algebra allows us to put this equation in a more convenient form:

$$f\left(x\right)=\left(\frac{500}{20000}\right)\left(\frac{50+jx}{10+jx}\right)\left(\frac{20+jx}{100+jx}\right) \tag{2.21}$$

The DC gain can be identified as $|f(0)|=0.025$. The zeros of the system are located at $x_{Z1}=20$ and $x_{Z2}=50$ while the poles are placed at $x_{P1}=10$ and $x_{P2}=100$. In decibels this function yields,

$$\left|f\left(x\right)\right|\left(dB\right)=-32\,dB+10*\log10\left(20^2+x^2\right)+10*\log10\left(50^2+x^2\right)-10*\log10\left(10^2+x^2\right)-10*\log10\left(100^2+x^2\right) \tag{2.22}$$

for x< 10, the magnitude is roughly constant, and equal to –32 dB; (0 zeros and 0 poles below x=10 give us an slope of 0 according to equation 2.18). Within the interval 10<x< 20, the gain decreases with a slope of –20 dB/decade due to the pole located at x=10 (1 pole below x=20). At x=20 we have the effect of the pole (-20 dB/decade) and the effect of the zero at x=20 which contributes with a slope of +20 dB/decade after this frequency. In the interval 20>x>50, the effect of the pole and zero cancel each other and the gain remains constant; in this interval we have 1 zero and 1 pole leading to a slope of 0. The zero located at x=50, comes into the picture for x> 50 adding an extra increment of +20 dB/decade. Finally at x>100, all poles and zeros determine the shape of the overall function. Since the number of poles (2) and zeros (2) is the same in this case, the slope of the transfer function for x > 100 is zero. The final value of the transfer function is obtained evaluating the original transfer function at $\omega=\infty$; for this example is it given by 10*50/(100*200)=0.025 (-32 dB). The magnitude of the transfer function is shown below; piece wise linear approximation is used.

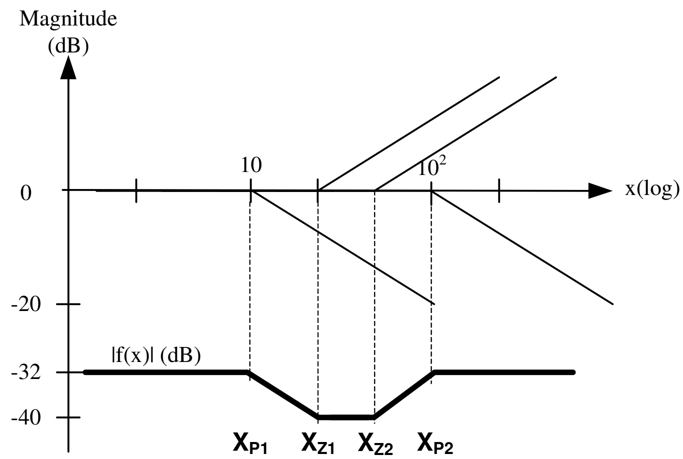


Fig. 2.8. Magnitude response of the complex function given by equation 2.21.

Let us now to consider the phase response of the previous example, equation 2.21. The phase of this function can be written as (please double check this result; do it!)

$$Phase(f(x)) = tan^{-1}\left(\frac{x}{20}\right) + tan^{-1}\left(\frac{x}{50}\right) - tan^{-1}\left(\frac{x}{10}\right) - tan^{-1}\left(\frac{x}{100}\right) \tag{2.23}$$

The piece-wise linear approximation for the phase response is shown in figure 2.9. The DC phase shift is obtained evaluating it from the original transfer function at x=0, 0 degrees in this case. After this frequency, we have to consider the phase contributions of the poles and zeros, and to add them together as follows. The pole located at x=10 introduces phase shift that goes from zero degrees at x-values below 1, -45 degrees at x=10 and –90 degrees at x>100; its effect can be approximated by a linear phase variation with a roll-off of -45 degrees/decade within the range 1<x<100. Similarly, the zero located at x=20 introduces 0 degrees for x<2, +45 degrees at x=20 and +90 degrees at x>200. The zero located at x=50 raises the phase at a rate of 45 degrees/decade starting at x=5 and ending at x=500. The last pole add a roll-off of -45 degrees/decade in the range 10<x<1000. The overall phase response is shown in Fig. 2.9. The phase response is sometimes obscure you must do more examples to master bode plots.

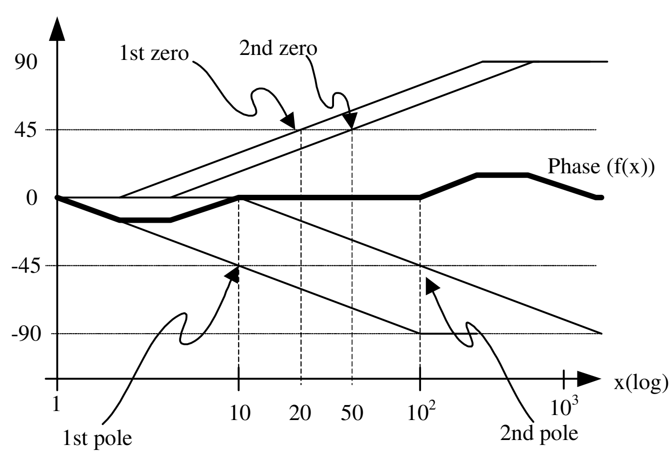


FIg. 2.9. Phase response approximation for the complex function given by equation 2.23.

II.3. Frequency response of first order systems.

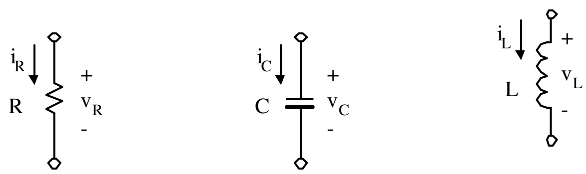
Steady-State Analysis: Resistive, Capacitive and inductive impedances. The voltage-current relationship for a linear resistor is determined by ohms law. Fortunately most of the resistors currently used in micro-electronics are fairly linear; hence it is easy to manipulate them using simple algebra:

$$v_R = R\, i_R \tag{2.24}$$

where v_R is the voltage drop across resistor while i_R is the current flowing through it. The voltage and current are defined in Figure 2.10a. The impedance (or commonly known as resistance) associated with the resistor is defined as the ratio of the voltage applied to its terminals divided by the current flowing through it; hence

$$\frac{v_R}{i_R} = Z_R = R \tag{2.25}$$

In this particular case the impedance is real and positive; therefore the voltage and current variables are in phase.
The current and voltage variables applied to a capacitor are related by the following fundamental relationship:



(a) (b) (c)

Fig. 2.10. a)Resistor, b) Capacitor and c) Inductor

$$i_c = C\left(\frac{dv_c}{dt}\right) \tag{2.26}$$

In this expression, C is the capacitance, usually measured in Farads; there are some circuits that show non-linear capacitance, however in this course we will consider that C remains constant (time voltage and current invariant) under all conditions of operation. ***If the AC voltage applied through the capacitive terminals has the general complex form $v_c=(V_C)e^{j\omega t}$***, hence the expression for the current flowing through can be computed from 2.26 as

$$i_c = j\omega C\, v_c \tag{2.27}$$

Thus, contrary to a resistor in which voltage and current are in phase, the voltage across the capacitor's plates and the current flowing through them are 90 degrees out of phase. Another remarkable property in 2.27 is that the current flowing between the capacitor plates increases proportional with the frequency of operation, leading to very large values at high frequencies even if the voltage variations are small. Based on expression 2.27, it is evident that the impedance of an ideal capacitor C is function of the frequency used, and it is given by

$$\frac{v_c}{i_c} = Z_c = \frac{1}{j\omega C} = \frac{1}{sC} \tag{2.28}$$

where ω (=$2\pi f$) is the frequency of the signal in radians/sec. Then impedance of a capacitor is negative and imaginary: =$1/j\omega C$. To simplify the algebra, usually the frequency variable is defined as **$s=j\omega$**; notice that **s is an imaginary variable**.
The magnitude of the capacitive impedance is infinite (open circuit) at ω=0; hence any DC voltage applied at its terminals of the capacitor does not generate any current in steady state. Some current is generated if the initial condition on the capacitor is different than the voltage you are applying, but as soon as the capacitor is charged to the applied dc level the current vanishes regardless of the magnitude of the applied DC voltage. It is also important to recognize that the capacitive impedance decreases with the frequency, and this impedance usually dominates the behavior of the electronic systems at very high frequencies. As an example, let us consider a capacitor of 100 pF

(10^{-10} F); the magnitude of the impedance at ω=1Krad/sec is 10^{7} Ω, but at 100Mrad/sec its impedance's magnitude drops down to 100 Ω, and at 1Grad/sec its equivalent impedance is only 10 Ω. In most linear circuits, with the exception of some special techniques such as switched-capacitor circuits, the effects of the capacitors in the range of few picofarads can be neglected at low frequencies (less than 1 MHz) because their impedance is usually very large. Before closing this section, it is important to remark that the voltage and current lumped to the capacitor are not in phase; the phase of the pure capacitive impedance is –90 degrees.

Following a similar procedure, the impedance of the inductor under steady state conditions can be easily found; the fundamental equations for the quantities defined in Figure 2.10c are given below. The voltage-current relationship for an ideal inductor when a sinusoidal signal is employed is

$$v_L = L\left(\frac{di_L}{dt}\right) \tag{2.29}$$

Where L is the inductance (in Henrys). If the inductance's current has the general complex form $i_L=I_L(e^{j\omega t})$, then the voltage across inductance's terminals yields,

$$v_L = j\omega L i_L \tag{2.30}$$

The voltage generated across the inductor and the current flowing through it are +90 degrees out of phase. The impedance of the inductor can be easily found from 2.30 as

$$\frac{v_L}{i_L} = Z_L = j\omega L = sL \tag{2.31}$$

Contrary to the case of the capacitor, the inductor presents very small impedance through its terminals at low frequency (zero at DC) and very high impedance (open circuit) at very high frequencies.

Voltage Divider. The voltage divider consists of two impedances connected in series; the output voltage is defined as the voltage drop across one of the elements used. The typical voltage divider is shown in figure 2.11; notice that the input voltage vi is divided in the impedance Z_1 and Z_2 and only one part of the input voltage define v_o. By using basic circuit theory (KVL) it can be easily shown that the voltage across the grounded impedance Z_2 is given by the following expression:

$$\frac{v_0}{v_i} = \frac{Z_2}{Z_1 + Z_2} \tag{2.32}$$

If the impedances Z_1 and Z_2 are of the same type (either real or imaginary), the amplitude of the output voltage v_0 is equal or smaller than the amplitude of the applied signal v_i; therefore, with this topology you loose some signal (attenuator). For that reason this circuit is known as voltage divider, since the input voltage gets split between the two impedances. In the most general case, Z_1 and Z_2 can be composed by several elements leading to a variety of transfer functions with different properties; the frequency response of some of these functions is analyzed in this section.

In order to get some insight on the circuit's behavior let us consider first the case of a simple resistive divider, as shown in figure 2.11b. The voltage gain in this case is

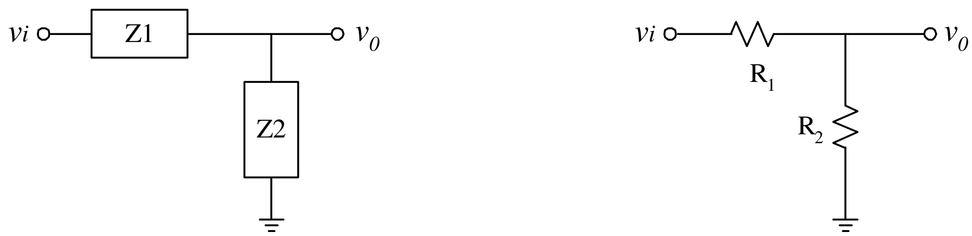


Fig. 2.11. Voltage divider: a) general scheme and b) resistive voltage divider.

$$\frac{v_0}{v_i} = \frac{R_2}{R_1 + R_2} \tag{2.33}$$

Since the resistors are ideally voltage and frequency independent elements, the voltage gain of the resistive voltage divider is a real number with magnitude less or equal than unity and phase shift of zero degrees for all frequencies. Therefore, the output signal is an attenuated replica of the incoming signal. Shown below is the magnitude response of the resistive voltage divider for different cases; in theory the circuit does not have any frequency limitation, but in practice several parasitic elements limit its frequency response. For these simulations, R1= 1kΩ and R2= {100, 1.1 kΩ and 2.1 kΩ}. The attenuation factors are 100/1100 (-20.8 dB), 1100/2100 (-5.6 dB) and 2100/3100 (-3.4 dB), respectively.

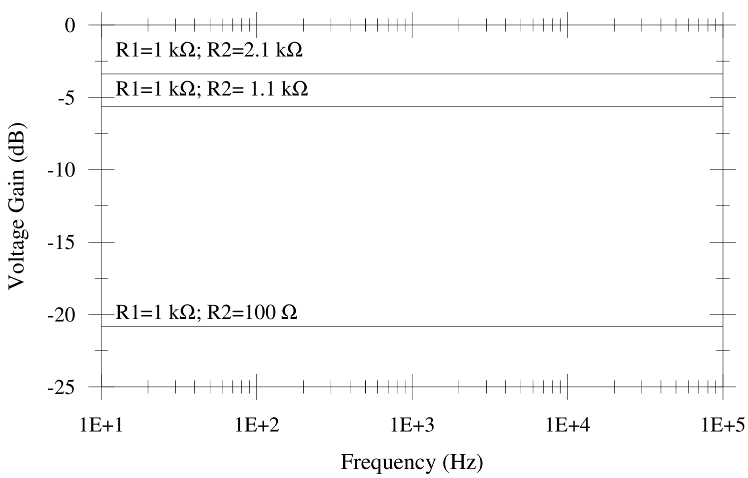


Fig. 2.12. Magnitude response of the resistive voltage divider.

It is important to emphasize that the attenuation factor reduces if the series resistance R1 is smaller than the grounded resistance R2 since the input voltage is split between the two impedances and the larger one takes most of the signal. In fact, if R_1 =0, this resistor does not generate any voltage drop and the voltage gain is unity (input signal is equal to output signal). **The output voltage (across Z_2) is the difference between the input voltage and the voltage drop through the series impedance Z_1**; hence **the larger the series resistance, the larger the voltage drop through this element is, and the smaller the output voltage. Notice that if resistors are used then**

$$v_0 = \frac{v_i}{1 + \frac{R_1}{R_2}} \tag{2.33b}$$

Therefore, the ratio of resistance values determines the output voltage.

First-Order low-pass transfer function. To get more insight on the properties of electronic networks using capacitors we analyze the voltage divider using capacitors and resistors. If Z_2 is replaced by a capacitor in the voltage divider of fig. 2.11, the circuit behaves as a low-pass filter. Lowpass filters are commonly used to suppress undesired high-frequency signal components and noise; in audio applications, the signal bandwidth is no more than

20 kHz, hence signals after that frequency should be suppressed before the information is converted into digital format for further processing and recording.
Using the expression of the capacitive impedance (eqn. 2.28) and equation 2.32, the voltage gain of the resulting first order system becomes

$$\frac{v_0}{v_i} = \frac{Z_2}{Z_1 + Z_2} = \frac{\frac{1}{j\omega C_2}}{R_1 + \frac{1}{j\omega C_2}} = \frac{\frac{1}{R_1 C_2}}{j\omega + \frac{1}{R_1 C_2}} = \frac{\frac{1}{R_1 C_2}}{s + \frac{1}{R_1 C_2}} \tag{2.34}$$

It is evident that due to the combination of the resistor (real impedance) and the capacitive (imaginary) impedance, the voltage gain becomes a complex function of the frequency variable ω. For very low frequencies ($\omega \sim 0$) the transfer function is positive and real (unity in this case). At the frequency of the pole (**$\omega=\omega_p=1/R_1C_2$**) the transfer function becomes equal to $\omega_p/(\omega_p + j\omega_p) = 1/(1+j)$; the magnitude of the transfer function is equal to 0.7071 (-3 dB). On the other hand, the phase shift between output and input is close to zero at very low frequencies and -45 degrees at $\omega=\omega_p$. For frequencies beyond $\omega_p=1/R_1C_2$ the transfer function is dominated by the factor $1/j\omega R_1C_2$ leading to a transfer function that decreases when the frequency increases. The transfer function reduces at high frequency because the capacitive impedance reduces at high frequency, and beyond ω_p the magnitude of the resistive impedance is greater than that of the capacitive impedance; hence most of the input signal appears through the larger impedance, in this case the resistor.
The regions of operation are summarized in the following equation:

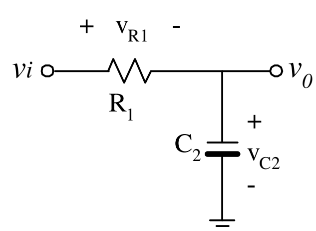


Fig. 2.13. First order lowpass filter

$$10\log 10\left(\left|\frac{v_0}{v_i}\right|^2\right) \cong \begin{cases} 0\,dB & \text{if} \quad \omega << \frac{1}{R_1C_2} \\ -3\,dB & \text{if} \quad \omega = \frac{1}{R_1C_2} \\ 20*\log 10\left(\frac{1}{R_1C_2}\right) - 20*\log 10(\omega) & \text{if} \quad \omega >> \frac{1}{R_1C_2} \end{cases} \tag{2.35}$$

The 2 regions of operation (flat response and stop-band) are clearly identified in the log-log plot depicted in figure 2.14a for two cases: $\omega_p = 1/R_1C_2 = 2\pi x300$ rad/sec and $\omega_p = 1/R_1C_2 = 2\pi x10000$ rad/sec. Notice that the ***pole's frequency** ω_p **is determined by the R_1C_2 product***.
Two regions of operation are clearly identified, and it is interesting to find a physical explanation to this behavior. If the first-order lowpass circuit is further analyzed we can noticed that for low frequencies, $\omega<<1/R_1C_2$, the series resistance R_1 is much smaller than the magnitude of the grounded impedance given by $1/\omega C_2$. Since the input voltage is split between the two elements then $v_{in}=v_{R1}+v_{C2}$, and if one of the impedances is much greater than the other one, most of the voltage will appear across the higher impedance. For low frequencies most of the input voltage is absorbed by the capacitor leading to $v_{C2}=v_{in}$ (unity gain and zero excess phase). ***At $\omega=1/R_1C_2$ the magnitude of both impedances are equal, and the input voltage is equality split between the two elements leading to $|v_{R1}|=|v_{C2}|=0.7v_{in}$ since the current flowing through both elements is the same but the resistance is real and the***

capacitive impedance is imaginary (**think about this!**). At high frequencies the impedance of the capacitor reduces further, then most of the input signal is absorbed by the series resistor; the higher the frequency the smaller the capacitive impedance, and the larger the attenuation factor is. Notice that at very high frequencies, the capacitive impedance is very small (zero, short circuit, at $\omega=\infty$) effectively shortening the output node to ground.

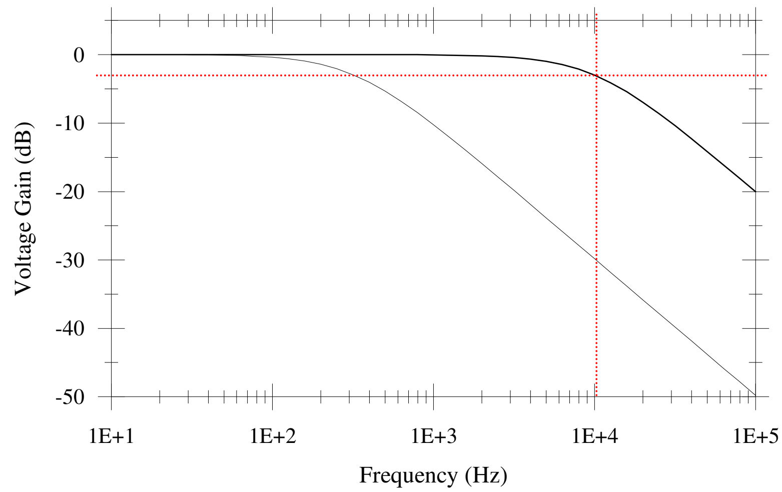


Fig. 2.14a. Magnitude response of the first order lowpass filter for two cases: $1/R_1C_2=2\pi x300$ and $1/R_1C_2=2\pi x10000$.

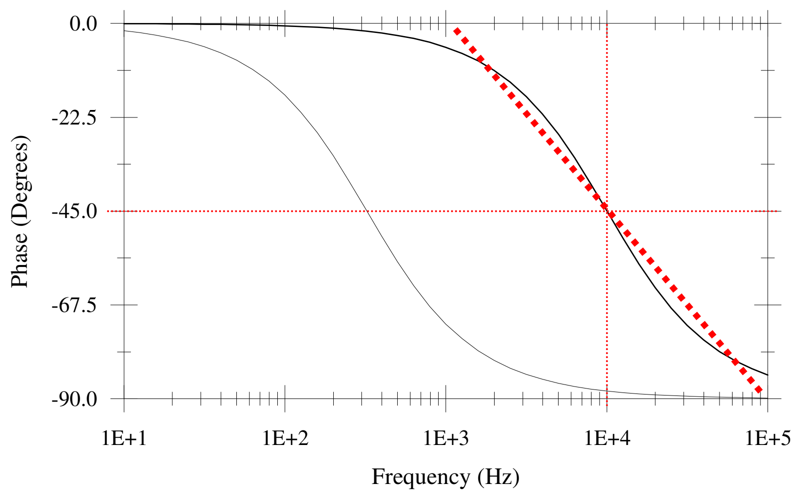


Fig. 2.14b. Phase response of the first order lowpass filter for the two cases.

The phase response (in radians) of the circuit's transfer function is obtained from 2.34 as

$$\text{Phase}\left(\frac{v_o}{v_i}\right)=\mathbf{tan}^{-1}\left(\frac{\frac{1}{R_1C_2}}{j\omega+\frac{1}{R_1C_2}}\right)=0-\mathbf{tan}^{-1}\left(\omega R_1C_2\right) \qquad (2.35)$$

The phase plot is depicted in Fig. 2.14b. As discussed in the previous subsection, the phase variation is mainly allocated within two decades around the pole's frequency; $0.1\omega_P < \omega < 10\omega_P$. For the case $\omega_P=1/R_1C_2=2\pi x10k$ rad/sec

(f_P=10kHz), the phase shift goes from 0 to -90 degrees within the frequency range of 1kHz-100kHz; the phase shift at f_P=10kHz is exactly -45 degrees.
The magnitude and phase plots are, of course, correlated with the time response of the circuit. Since the system is linear, and if a sinusoidal signal is applied at the input of the lowpass filter, the output will be a sinusoidal function as well but with different amplitude and different phase. In case the frequency of the input signal is below the pole's frequency, the amplitude of the output signal is very close to the amplitude of the input signal and the phase difference is small as well, as shown in figure 2.15a. At the pole's frequency the amplitude of the output is 0.7 v_{in} and the phase lag is -45 degrees. For frequencies beyond 10 times the pole's frequency, the phase lag is close to -90 degrees, and the magnitude of the output signal is very small as depicted in the Fig. 2.15c.

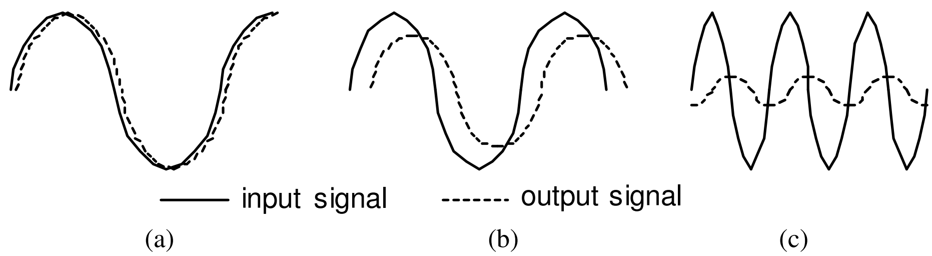


Fig. 2.15. Signals present at the input (continuous curve) and output (dashed curve) of the first order lowpass filter: a) $\omega<\omega_p$, b) $\omega=\omega_p$ (magnitude of the putput signal is 0.7*Vin and phase shift is –45 degrees) and c) $\omega>\omega_p$ (output signal is small and phase shift is close to –90 degrees)

Before we continue plotting the transfer functions and using BODE approximations it is desirable to introduce another useful parameter: the circuit's time constant. The first order low-pass filter is described by equation 2.34; if the s variable is used instead of jω, pole is located at

$$S_p = -\frac{1}{R_1C_2} \tag{2.36}$$

S_p is a left-hand plane pole in the complex s-plane; it will be evident soon that it is convenient to use $\omega_p=1/R_1C_2$. ***The circuit's time constant is defined as the product R_1C_2; the unit of the RC products is time (seconds) indeed.*** In general, if the circuits are composed by active elements, capacitors and resistors, the frequency of both poles and zeros are determined by equivalent RC products (time constants).

High-pass transfer function. Another interesting circuit is the so-called first order high-pass filter. This is obtained if a capacitor is connected in series with a resistor as shown in figure 21.6. This configuration is found in most of the AC-coupled amplifiers. By using basic circuit analysis the transfer function can be obtained as:

$$\frac{v_0}{v_i} = \frac{R_2}{R_2 + \frac{1}{sC_1}} = \frac{s}{s + \frac{1}{R_2C_1}} \tag{2.37}$$

where s=jω. The zero is located at ω=0 and the pole is placed at $\omega_p=1/R_2C_1$. As aforementioned a zero increases the magnitude at a rate of + 20 dB/decade while a pole decreases it at the same rate. As a result, three regions can be identified when plotting the magnitude response: a) magnitude behavior for $\omega < \omega_p$; b) magnitude of the transfer function at $\omega = \omega_p$; and c) magnitude of the transfer function at $\omega > \omega_p$. The fundamental equations for the square of the magnitude are given in the following expressions

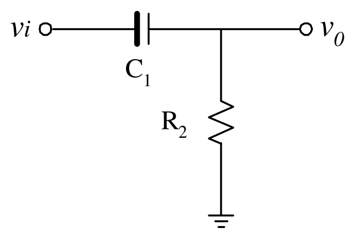


Fig. 2.16. First-Order high-pass filter.

$$\left|\frac{v_0}{v_i}\right|^2 = \begin{cases} (\omega R_2 C_1)^2 & \text{if } \omega << \dfrac{1}{R_2C_1} \\ 0.5 & \text{if } \omega = \dfrac{1}{R_2C_1} \\ 1 & \text{if } \omega >> \dfrac{1}{R_2C_1} \end{cases} \tag{2.38a}$$

or in decibels

$$\left|\frac{v_0}{v_i}\right|_{dB} \cong \begin{cases} 20 * \log 10(R_2C_1) + 20 * \log 10(\omega) & \text{if } \omega << \dfrac{1}{R_2C_1} \\ -3 & \text{if } \omega = \dfrac{1}{R_2C_1} \\ 0 & \text{if } \omega >> \dfrac{1}{R_2C_1} \end{cases} \tag{2.38b}$$

The transfer function starts at zero (-∞ in dB), and increases at a rate of +20 dB/decade when the frequency is swept until the pole's frequency; at higher frequencies the output voltage is in phase with the input signal and the amplitude of both signals becomes very similar.

Regardless how the transfer function looks like, it is important to notice that the capacitive impedance is extremely large at DC and low frequencies, hence most of the input signal is absorbed by this element; therefore the voltage drop across the resistor is very small at low frequencies. At very high frequencies, the capacitive impedance decreases, and could be even consider as zero for fast computations; in this case all the input signal is absorbed by the resistor leading to unity voltage gain. The magnitude and phase responses of the filter are depicted in the following plots for two cases: $\omega_p = 1/R_1C_2 = 2\pi x300$ rad/sec and $\omega_p = 1/R_1C_2 = 2\pi x10000$ rad/sec.

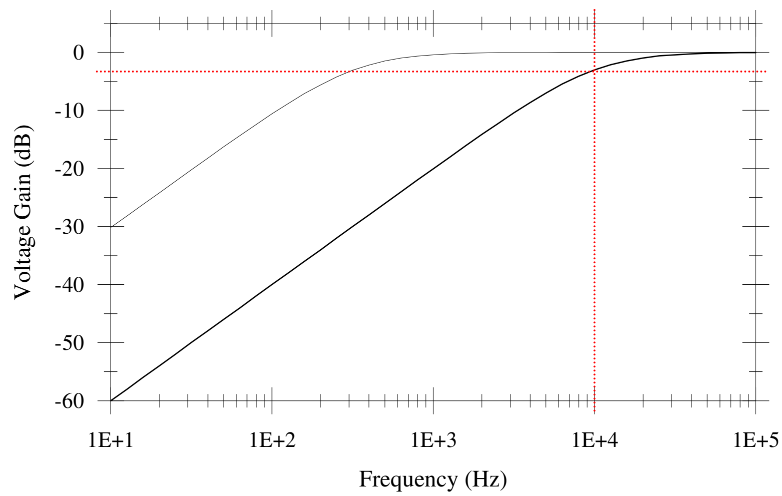


Fig. 2.17a. Magnitude response of the first order highpass filter for two cases: f_p =300Hz and f_p =10kHz.

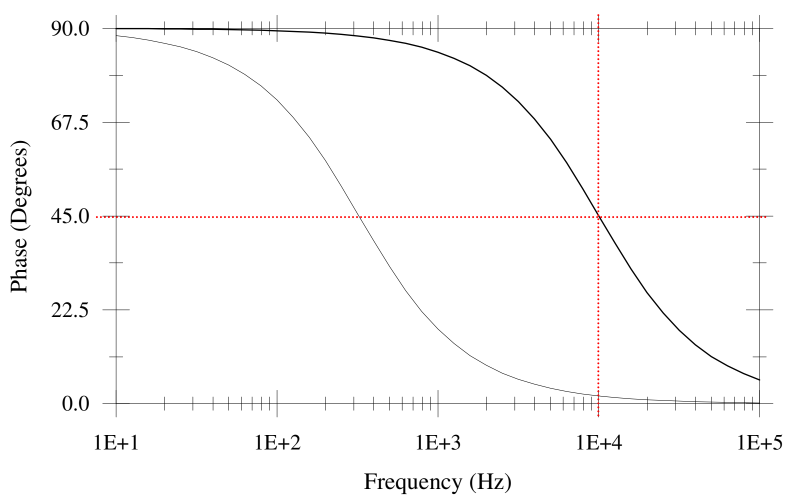


Fig. 2.17b. Phase response of the first order highpass filter.

General first-order transfer function. The most general first order filter presents a zero and a pole; the circuit is depicted in fig. 2.18. Regarless where the zero-pole is located, the circuit can be analyzed by inspection:

1. At very low frequencies the capacitors can be considered as open circuits, leading to a low-frequency gain given by $R_2/(R_1+R_2)$. The phase shit is zero degrees.
2. For very high frequencies, the capacitors dominate the impedance, hence the high-frequency gain is given by $C_1/(C_1+C_2)$. The phase shift is also 0 degrees.
3. If the low-frequency gain is smaller than the high-frequency gain, then the frequency of the zero is at lower frequencies than the pole's frequency since the transfer function must increase.
4. If the low-frequency gain is greater than the high-frequency gain, then the frequency of the pole is at lower frequencies than the zero's frequency since the transfer function must decrease.

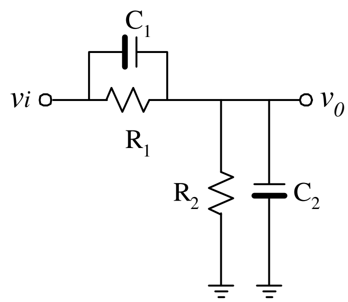


Fig. 2.18. General first order filter.

The circuit transfer function can be obtained as (**find it yourself!**):

$$\frac{v_0}{v_i}=\left(\frac{C_1}{C_1+C_2}\right)\left(\frac{s+\dfrac{1}{R_1C_1}}{s+\dfrac{1}{\dfrac{R_1R_2}{R_1+R_2}\left(C_1+C_2\right)}}\right) \tag{2.39}$$

The low-frequency gain can be obtained replacing the capacitors by an open circuit, and solving the resulting circuit; it is given by

$$\left.\frac{v_0}{v_i}\right|_{\omega=0}=\frac{R_2}{R_1+R_2} \tag{2.40}$$

The gain at the mid-band depends on the location of the pole and zero. The frequency of the zero and pole are given by

$$\omega_z=\frac{1}{R_1C_1}$$
$$\omega_p=\frac{1}{\dfrac{R_1R_2}{R_1+R_2}\left(C_1+C_2\right)} \tag{2.41}$$

And the magnitude of the transfer function at high frequencies is obtained evaluating the transfer function for $\omega=\infty$; it leads to the following result

$$\left.\frac{v_0}{v_i}\right|_{\omega=\infty}=\frac{C_1}{C_1+C_2} \tag{2.42}$$

The magnitude and phase response are depicted in the following plots for the cases:

i) C_1= 0.159 μF, C_2= 1.59 μF, R_1= R_2=100Ω

ii) C_1= 0.159 μF, C_2= 1.59 μF, R_1= R_2=4.1 kΩ

For the first case, the dc gain is 1/2 (-6 dB) and the high frequency gain is 0.0909 (-20.8 dB). The frequency of the zero is ω_z=62.8 krad/sec (=10 kHz) and the pole is located at ω_p=11.435 krad/sec (=1.82 kHz). Piecewise linear approximations can be easily plotted for both magnitude and phase if you follow the rules discussed in previous sections.

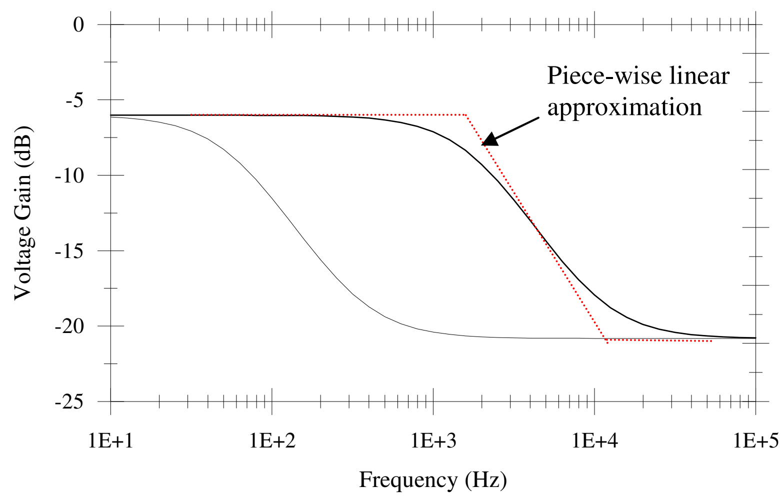


Fig. 2.19. Magnitude response of the first order filter.

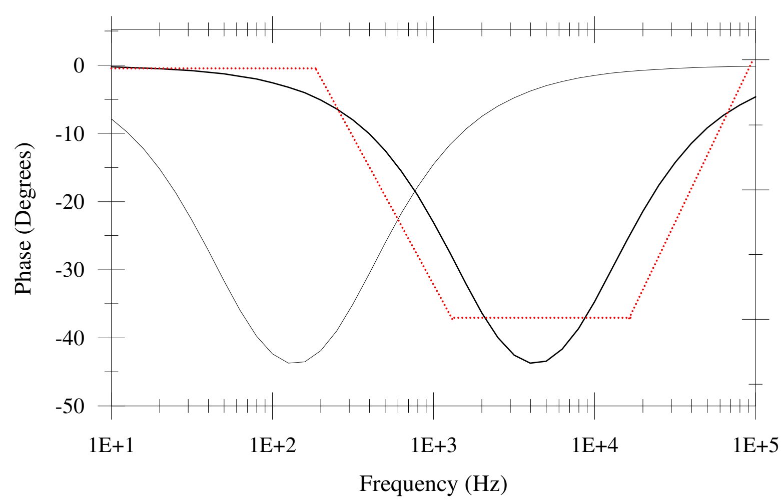


Fig. 2.20. Phase response of the first order filter.

II.4. The current divider.

The current divider is found when the input and output signals are current; many important integrated circuit realizations are based on current because the signal at the collector (in bipolar junction transistors) terminal is current. The basic structure of the current divider is depicted in the following figure.

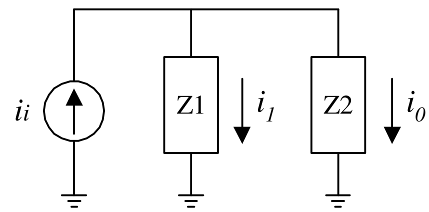


Fig. 2.21. Current divider.

The problem here is to find the output current i_0 as function of the input signal and the associated impedances. Certainly the output current depends on the relationship between the impedances Z1 and Z2. Using conventional circuit analysis techniques allow us to find the current gain as:

$$\frac{i_o}{i_i} = \frac{Z_1}{Z_1 + Z_2} \tag{2.43a}$$

The current flowing through Z2 increases with increment (or decrement) in Z1 (Z2). An important remark here is: most of the current flow through the smaller impedance. Remember that $i_i = i_1 + i_2$, hence if more current flows through one of the impedances, lesser current flows through the other one such that the addition remains equal to the input current. Another similarity with the voltage divider is that the current gain is function of the relative values of the impedances rather than the absolute value of the impedance, as clarified in the following expression.

$$\frac{i_o}{i_i} = \frac{1}{1 + Z_2/Z_1} \tag{2.43b}$$

For a resistive current divider (Z1=R1, Z2=R2) it can be seen that the output current is further reduced if R2>>R1; most of the incoming current flows through the smaller impedance (R1) and very little current is collected at the output. On the other hand, most of the current flows through Z2 (or R2) if R2<< R1.
In the case of a combination of capacitor-resistor, similarly to the voltage divider, the current transfer function is a frequency dependent complex function. A lowpass transfer function is obtained if the configuration shown in Fig. 2.23 is used. Using 2.43, it follows

$$\frac{i_o}{i_i} = \frac{\dfrac{1}{j\omega C_1}}{\dfrac{1}{j\omega C_1} + R_2} = \frac{\dfrac{1}{R_2 C_1}}{s + \dfrac{1}{R_2 C_1}} \tag{2.44}$$

At low frequencies the capacitive impedance is much greater than R2, and most of the current flows through the resistor; as a result the current gain is close to unity. At $\omega = 1/R_2C_1$ ($R_2 = |1/j\omega C_1|$) the magnitude of the current gain is equal to $1/2^{1/2}$, and for higher frequencies the current gain reduces inversely proportional to ω. This structure is useful for the analysis of multistage and cascode amplifiers. The magnitude and phase plots of the current gain are similar to the ones obtained for the voltage divider and are not shown here, but we encourage the reader to plot them.

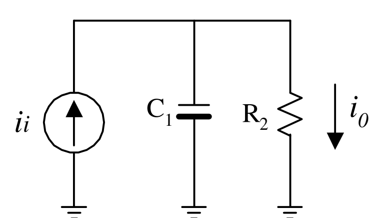


Fig. 2.23. RC-current divider.

It is interesting to notice that the current flowing through the capacitor C_1, i_{C1}, has the high-pass behavior; this can be easily understood by observing that the addition of i_{C1} and i_0 must be equal to i_i. Since i_0 reduces at high frequencies, the current flowing through C_1 must increase to maintain the overall current equal to the input current. i_{C1} is given by the following expression:

$$\frac{i_{C1}}{i_i}=\frac{i_i-i_0}{i_i}=\frac{R_2}{\frac{1}{j\omega C_1}+R_2}=\frac{j\omega R_2C_1}{1+j\omega R_2C_1} \qquad (2.45)$$

The concept of time constant. From the previous examples, it is clear that the Resistor-Capacitor (RC) products define the frequencies of poles and zeros of the transfer function. The RC product is also termed circuit's time constant τ; poles and zero's frequencies (in radians/second) are usually defined by $1/\tau$. Finding the time constants lumped to the nodes allow us to identify the poles and zeros. We will learn more about circuit analysis by inspection in the following sections.

II.5. First order system: Time response.

If the frequency response of a circuit is known, the time response can also be obtained by using the properties of the Laplace transform which relates the time and frequency domains. Time domain analysis will be considered in the following chapters, but it is sufficient to say now that frequency domain analysis is simpler to use especially for complex applications under steady state conditions. These techniques allow us to get insight on the behavior of the circuit, and easy to extrapolate the results to cases in which the input signal is smooth (no jumps, spikes or square waves). For that reason sinusoidal signals are often used in frequency analysis.

If f(t) is a function defined for t>0, the Laplace transform is denoted as F(s) and defined as

$$F(s)=\ell(f(t))=\int_0^{\infty}e^{-st}f(t)dt \qquad (2.46)$$

where $s=j\omega$. The laplace transform of f(t) exists if and only if equation 2.46 converges to a finite value for all values of s. Some important properties of the laplace transform are listed in the following table:

$$\ell(C_1 f_1(t) + C_2 f_2(t)) = C_1 \ell(f_1(t)) + C_2 \ell(f_2(t)) = C_1 F_1(s) + C_2 F_2(s)$$

$$\ell\left(e^{at} f(t)\right) = F(s-a)$$

$$\ell(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$\ell(f'(t)) = sF(s) - F(0)$$

$$\ell(f''(t)) = s^2 F(s) - sF(0) - F'(0)$$

$$\ell\left(\int_0^t f(x)dx\right) = \frac{F(s)}{s}$$

$$\ell\left(t^n f(t)\right) = (-1)^n \frac{d^n}{ds^n} f(s)$$

Table 2.3. Properties of the Laplace transform.

Among other important properties of laplace transforms not discussed in these notes we have to consider the initial and final value theorems, impulse and pulse transformations, and system response for periodic functions. Please refer to any text book on laplace transforms to learn more about these topics.

The Laplace transform of some important functions are given in the following table:

F(s)	F(t)
1	$\delta(t)$
e^{-as}	$\delta(t-a)$
$\frac{e^{-as}}{s}$	u(t-a)
$\frac{1}{s}$	1 t>=0 (step)
$\frac{1}{s^n}$	$t^{n-1}/(n-1)!$
$\frac{1}{(s-a)^n}$	$t^{n-1}e^{at}/(n-1)!$
$\frac{1}{(s-a)(s-b)}$ a≠b	$\frac{e^{bt}-e^{at}}{b-a}$
$\frac{s}{(s-a)(s-b)}$ a≠b	$\frac{be^{bt}-ae^{at}}{b-a}$

Table 2.4. Laplace transform of some relevant transfer functions.

To illustrate how to relate the time domain analysis and the s-domain transfer function, let us consider the following s-domain transfer function

$$\frac{v_o(s)}{v_i(s)}=\frac{s^m+a_{m-1}s^{m-1}+....+a_1s+a_0}{s^n+b_{n-1}s^{n-1}+....+b_1s+b_0} \tag{2.47}$$

As illustrated in Fig. 2.24 for a first order system, any system can be characterized either in frequency (magnitude and phase of the input and output signals as well as determining the impulse response of the circuit; e.g. making vi(t)=δ(t) and finding $v_o(t)=h(t)\delta(t)$ where $h(t)=\ell^{-1}H(\omega)$. System response for any input signal can be obtained by taking the convolution of the impulse response of the circuit h(t) with the input signal. Another reason why impulse response is frequently used for the time-domain characterization is because it can be easily obtained from the s-domain transfer function, especially because there are plenty of tables available in the open literature.

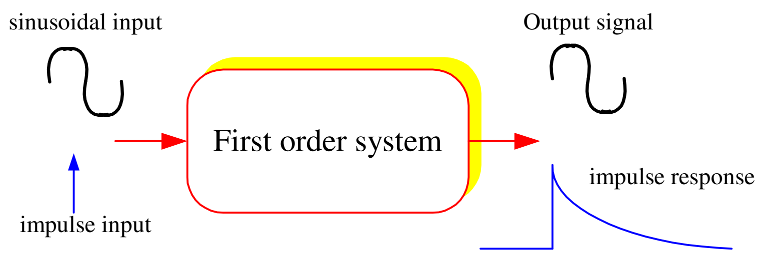


Figure 2.25. System characterization using sinusoidal functions (frequency domain) and impulse response (time domain).

The impulse input signal in the time domain corresponds to $v_i(s)=\ell(\delta(t))=1$, according to the properties of laplace transforms, see first property in the previous table. Hence, the time-domain impulse response of 2.47 is computed as follows:

$$v_o(t)=\ell^{-1}\left(\frac{s^m+a_{m-1}s^{m-1}+....+a_1s+a_0}{s^n+b_{n-1}s^{n-1}+....+b_1s+b_0}\right) \tag{2.48}$$

Therefore, the **laplace transform of the overall transfer function corresponds to system's impulse response in the time domain**.

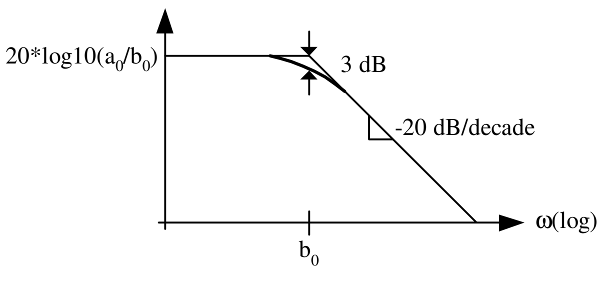
Example: Impulse response. Let's find the impulse and pulse response of a first order single-pole transfer function. If the s-domain transfer function to be consider is the following function

$$v_o(s)=\frac{a_0}{s+b_0}v_i(s) \tag{2.49}$$

Then, the impulse response is obtained by finding the inverse laplace transform of this equation. From the previous table it follows that the impulse response ($v_i(s)=1$) of the system should be computed as

$$v_o(t)=\ell^{-1}\left(\frac{a_0}{s+b_0}\right)=a_0e^{-b_0t} \tag{2.50}$$

This expression corresponds to the typical exponential function that results from the solution of first-order systems. The magnitude response (frequency response) of the first order filter is depicted in Figure 2.25a. The DC gain is given by a_0/b_0 while the –3dB frequency is determined by b_0. The corresponding unity impulse response is shown in figure 2.25b.



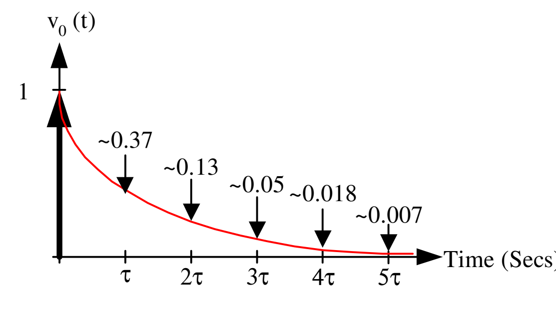


Fig. 2.25 Magnitude response of a first-order lowpass system and b) unity impulse response. **τ=1/b_0.**

In these results, it is assumed that the initial conditions of the first order system are zero ($v_0(0)=0$). When the input impulse is applied, the steady state of the system is disturbed; the output voltage jumps to the amplitude of the input impulse, in this case 1. The system returns back to its steady state according to an exponential decaying behavior. The system's time constant time τ (=1/b_0) is determined by the pole's frequency b_0. After 1 time-constant $t=\tau$ the output function decays 63%, and for $t=2\tau=2/b_0$ the output voltage decays 87% from its original value. After $t=4\tau$, the output is very close to its final value; deviation from its final value is less than 2%. For many applications, settling errors below 1% are acceptable; hence after the impulse is applied, we have to wait at least 5 time constants before the system returns back to its steady state; zero voltage in the particular case of a impulse applied to the input. **Example: Step response.** The step response of a system characterized by 2.43 can easily be obtained if the second last property given in table 2.3 is used. Since the step function in the s-domain correspond to 1/s in the frequency response, then the time-domain output response to a unity input step yields

$$v_o(t)\big|_{step} = \int_0^t v_o(t)\big|_{impulse} dt = \frac{a_0}{b_0}\left(1-e^{-b_0 t}\right)V_{step} + v_0(0) \qquad (2.51)$$

where $V_o(t)|_{impulse}$ is given by equation 2.50, V_{step} is the amplitude of the applied step and $v_0(0)$ is the value of the output voltage at t=0. The step response of the circuit presents an exponential behavior. If the initial conditions are zero, then the final output voltage (value at $t=\infty$) is given by $(a_0/b_0)V_{step}$. After 1 time constant, the final value is around 63% of its final value, and around 99.3 % after 5 time constants, as shown in figure 2.26. The time to reach 99% of its final value is often called 1 % (error) settling time, and the time required to reach this condition is approximated as $t=5\tau = 5/b_0$. Thus, fast settling behavior implies small time constants (RC products) or equivalently high pole's frequency. The **smaller the RC product the faster the system** is.

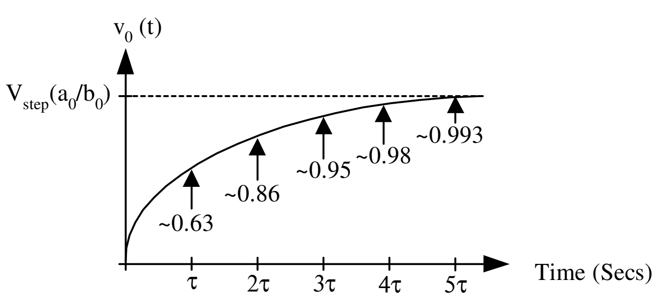


Fig. 2.26 Pulse response for the first order system defined by equation 2.49.

General case.
The general transfer function given in 2.47 can also be expressed as

$$v_o(s)\left(s^n + b_{n-1}s^{n-1} + ... + b_1 s + b_0\right) = v_i(s)\left(s^m + a_{m-1}s^{m-1} + ... + a_1 s + a_0\right) \tag{2.52}$$

Applying the laplace transform to this expression and making the initial conditions equal zero, we can obtain the time-domain input-output relationship as follows:

$$\begin{aligned}\frac{d^n v_o(t)}{dt^n} + b_{n-1}\frac{d^{n-1}v_o(t)}{dt^{n-1}} + ... + b_1\frac{d^1 v_o(t)}{dt^1} + b_0 v_o(t) = \\ = \frac{d^m v_i(t)}{dt^m} + a_{m-1}\frac{d^{m-1}v_i(t)}{dt^{m-1}} + ... + a_1\frac{d^1 v_i(t)}{dt^1} + a_0 v_i(t)\end{aligned} \tag{2.53}$$

The time domain output can be obtained by solving the previous differential equation for a given input. Usually the solution of this equation is not trivial, remember your math courses! It is a lot easier to solve the circuit in the frequency domain; find the output voltage and using the inverse laplace transform you can obtain system's time response.

3rd example: Find the impulse response of the system described by the following second order s-domain transfer function:

$$v_o(s) = \frac{10}{\left(s + (10 + j10)\right)\left(s + (10 - j10)\right)} v_i(s) \tag{2.54}$$

The poles of this function are complex conjugate; it can easily find that the poles are given by

$$s_p = \alpha \pm j\beta = -10 \mp j10 \tag{2.55}$$

The impulse response can be obtained by using the results of table 2.4 for the laplace transform of a second order function with two different poles. The result is

$$v_o(t) = \ell^{-1}\left(\frac{k}{(s - s_{p1})(s - s_{p2})}\right) = k\left(\frac{e^{s_{p2}t} - e^{s_{p1}t}}{s_{p2} - s_{p1}}\right) \tag{2.56}$$

If the poles are complex conjugate given by equation 2.55, the system's output voltage given by 2.56 becomes:

$$v_o(t) = k\left(\frac{e^{(\alpha - j\beta)t} - e^{(\alpha + j\beta)t}}{-j2\beta}\right) = ke^{\alpha t}\left(\frac{e^{j\beta t} - e^{-j\beta t}}{j2\beta}\right) = \left(ke^{\alpha t}\right)\left(\sin(\beta t)\right) \tag{2.57}$$

By using the numerical values we finally get

$$v_o(t) = \left(10e^{-10t}\right)\left(\sin(10t)\right)$$

The impulse response corresponds to a sinusoidal function with the oscillating frequency (in radians per second) given by the imaginary part of the complex conjugate poles. The amplitude of the sinusoidal function is modulated by an exponential function, with the exponent being determined by the pole's real part. A typical impulse response is depicted in the following figure.

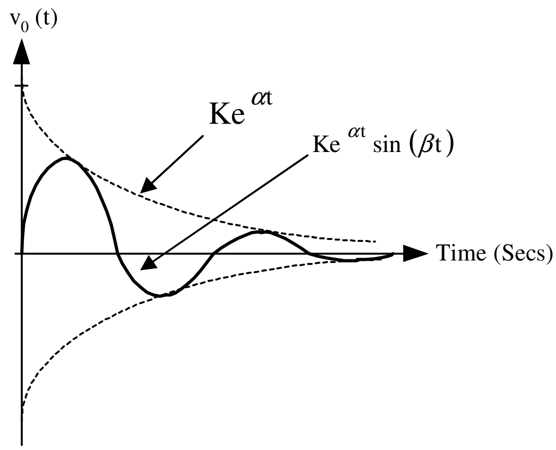


Fig. 2.27. Typical pulse response of a second-order transfer function. Notice that $\alpha<0$.

Notice that the impulse response of a **second-order system with poles located on the left hand side of the s-plane ($\alpha<0$) eventually returns back to its steady state (zero output voltage)**. Impulse response of systems with poles located at the right hand side of the s-plane lead to a positive exponent and the impulse response diverges to +/- ∞, making the **system unstable. A system is unstable if its output is unbounded as a result of a bounded input**! A necessary condition for system stability is to have system poles located on the left hand side of the complex s-plane.