**Introduction to Per Unit Calculations (ECE 459 note)**

Observations: With transformers, we need to convert the voltage magnitudes, impedances, currents, etc. repeatedly.

Questions: Can we bypass these complications?

Answer: By using per unit calculations that different voltage bases (but the same power base) are used in different transformer sections. The conversions among transformer voltages are absorbed by the base relationships. Thus p.u. representations of system elements (transformers, impedances, loads, generations, etc.) become more uniform and the fixed-tap transformers disappear from the p.u. representation. The same situations apply to wye-delta circuits, and they will have the same p.u. representation. Again, the conversion is absorbed into the base relationship. The same for single-phase and three-phase representations.

Consider the simple circuit of Figure 1 in which a load impedance of

\[ \tilde{Z}_L = 60 + j70 \, \Omega = 92.2 \angle 49^\circ \Omega \]

is connected to a voltage source. The no load voltage of the source is \( \tilde{E}_S = 1000 \angle 0^\circ \). The internal resistance of the source is \( \tilde{Z}_S = j10 \, \Omega \). The total impedance of the circuit is \( \tilde{Z}_T = 60 + j80 \, \Omega \). The load current through the circuit is

\[ \tilde{I}_L = \frac{\tilde{E}_S}{\tilde{Z}_T} = \frac{1000 \angle 0^\circ}{60 + j80} = 10 \angle -53^\circ \]

The apparent power produced by the source is

\[ \tilde{S}_S = \tilde{E}_S \cdot \tilde{I}_L^* = 1000 \angle 0^\circ \cdot 10 \angle 53^\circ = 10,000 \, \text{VA} \angle 53^\circ \]

In this circuit, we will designate the **magnitude** of each of these values as the reference value for that quantity, i.e. the reference value will be one unit of that quantity. That is,
1000 V is equal to one unit for voltage \((V_{\text{base}} = 1000 \text{ V})\)

100 Ω is equal to one unit for resistance \((Z_{\text{base}} = 100 \text{ Ω})\)

10 A is equal to one unit for current \((I_{\text{base}} = 10 \text{ A})\)

10,000 VA is equal to one unit for power \((S_{\text{base}} = 10,000 \text{ VA})\)

These basic units are called the **base values**. Therefore, the base values for our circuit are 1000 V, 100 Ω, 10 A, and 10 kVA.

Now we can refer any value of any quantity in the circuit to these base values. For example, the magnitude of the voltage across the load is

\[
V_L = E_S \frac{Z_L}{Z_T} = 1000 \cdot \frac{92.2}{100} = 922 \text{ V}
\]

In terms of our chosen base voltage, the load voltage is the ratio of the actual voltage and the chosen base voltage, i.e. 
\[
\frac{V_L}{V_{\text{base}}} = \frac{922}{1000} = 0.922
\] of the chosen base voltage. We say that the voltage is 0.922 per unit of volts. The designation is \(V_{\text{L.p.u.}} = 0.922 \text{ p.u. (V)}\). We can also say that the load voltage is 92.2% of the base voltage.

Similarly, in terms of our impedance base, the load impedance is

\[
Z_{\text{L.p.u.}} = \frac{Z_L}{Z_{\text{base}}} = \frac{92.2}{100} = 0.922 \text{ p.u. (Ω)}
\]

Just like the magnitude of the load impedance, we can express any ohmic value in terms of the base impedance. So the per unit resistance of the load is

\[
R_{\text{L.p.u.}} = \frac{R_L}{Z_{\text{base}}} = \frac{60}{100} = 0.6 \text{ p.u. (Ω)}
\]

The per unit reactance of the load is

\[
X_{\text{L.p.u.}} = \frac{X_L}{Z_{\text{base}}} = \frac{70}{100} = 0.7 \text{ p.u. (Ω)}
\]

Notice here that both, the resistance and the reactance are compared to base impedance.

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1 \(\bar{I}_L^*\) is the conjugate of the load current. In this case \(\bar{I}_L^* = 10A \angle 53^\circ\)
The power dissipated by the resistance is \( P = RI^2 = 60 \cdot 10^2 = 6000 \text{W} \). Expressed as per unit value on our base of 10,000 VA, the dissipated power is

\[
P_{\text{p.u.}} = \frac{P}{S_{\text{base}}} = \frac{6000}{10,000} = 0.6 \text{p.u. (W)}
\]

The reactive power to the load is \( Q_L = X_L I^2 = 70 \cdot 10^2 = 7000 \text{ VAr (inductance).} \)

Expressed as per unit value on our base of 10,000 VA, the load reactive power is

\[
Q_{L,\text{p.u.}} = \frac{Q_L}{S_{\text{base}}} = \frac{7000}{10000} = 0.7 \text{ p.u. (VAr) (ind)}
\]

Similarly as in handling the per unit values of ohmic quantities, both real power and reactive power are compared to the apparent base power, usually called base MVA. **Note that the p.u. values are much more uniform than the original values!**

All base values are only magnitude. They are not associated with any angle. The per unit values, however, are phasors. The phase angles of the currents and voltages and the power factor of the circuit are not affected by the conversion to per unit values. The per unit values can be written as phasors, and all calculations that can be done with the actual quantities can be also done with per unit values. For example, in our circuit, the per unit load impedance and the per unit load current are:

\[
\vec{Z}_{L,\text{p.u.}} = 0.922 \angle 49^\circ \text{ p.u. (}\Omega\text{)} \quad \text{and} \quad \vec{I}_L = 1 \angle -53^\circ \text{ p.u. (A)}
\]

Thus the voltage across the load is

\[
\vec{V}_{L,\text{p.u.}} = \vec{Z}_{L,\text{p.u.}} \cdot \vec{I}_{L,\text{p.u.}} = 0.922 \angle 49^\circ \cdot 1 \angle -53^\circ = 0.922 \angle -4^\circ \text{ p.u. (V)}
\]

In general, the per unit value is the ratio of the actual value and the base value of the same quantity.

\[
\text{per unit value} = \frac{\text{actual value}}{\text{base value}}
\]
Manufacturers give impedance of equipment in percent on own base. The percent value is the per unit value multiplied by 100: \( Z_{\%} = Z_{\text{p.u.}} \times 100\% \). The expression “own base” means that the base voltage is the rated voltage of the equipment, and the base power is the rated apparent power (in VA) of the equipment. The base current and the base impedance are calculated from the base voltage and the base VA:

\[
I_{\text{base}} = \frac{S_{\text{base}}}{V_{\text{base}}} \quad \text{and} \quad Z_{\text{base}} = \frac{V_{\text{base}}^2}{S_{\text{base}}}
\]

The percent impedance is then

\[
Z_{\%} = \frac{Z_{\text{actual in } \Omega}}{Z_{\text{base in } \Omega}} \times 100\%
\]

**Generators and Per Unit System**

A generator rated 1000 VA and 200 V has internal impedance of \( j10 \Omega \).

![Figure 2 – Basic Circuit: Example of p.u. calculations for generators](image)

The ratings of the generator are chosen as the base values. In this example,

\[
\begin{align*}
S_{\text{base}} &= 1000 \text{ VA} \\
V_{\text{base}} &= 200 \text{ V} \\
I_{\text{base}} &= \frac{S_{\text{base}}}{V_{\text{base}}} = \frac{1000}{200} = 5 \text{ A} \\
Z_{\text{base}} &= \frac{V_{\text{base}}^2}{S_{\text{base}}} = \frac{200^2}{1000} = 40 \Omega 
\end{align*}
\]

Note that the magnitude order differences are absorbed in the base!
The generator impedance is stamped on the nameplate together with the other ratings. This generator has impedance of j25%. This means that the per unit impedance is

$$Z_{\text{p.u.}} = \frac{Z_{\text{sh}}}{100} = \frac{j25}{100} = j0.25 \ \text{p.u.} \ (\Omega)$$

The actual impedance in Ω is $Z_{\text{actual}} = Z_{\text{p.u.}} \times Z_{\text{base}} = j0.25 \times 40 = j10 \ \Omega$

**Example:** The generator above is short circuited at its terminals. Find the short circuit current and the short circuit power delivered by the generator in p.u., in %, and in the actual units.

**Solution:**

![Diagram of generator with short circuit](https://via.placeholder.com/150)

$$\bar{I}_{\text{sc.p.u.}} = \frac{\bar{E}_{\text{G.p.u.}}}{\bar{Z}_{\text{S.p.u.}}} = \frac{1}{j0.25} = -j4.0 \ \text{p.u.} \ (A)$$

$$\bar{I}_{\text{sc} \ %} = \bar{I}_{\text{sc.p.u.}} \times 100\% = -j400 \ %$$

$$\bar{S}_{\text{sc.p.u.}} = \bar{E}_{\text{G.p.u.}} \times \bar{I}_{\text{sc.p.u.}} = 1 \times (-j4)^* = j4.0 \ \text{p.u.} \ (A)$$

$$\bar{S}_{\text{sc} \ %} = \bar{S}_{\text{sc.p.u.}} \times 100\% = j400 \ %$$

The actual values calculated from the per unit values are:

$$\bar{I}_{\text{sc}} = \bar{I}_{\text{sc.p.u.}} \times I_{\text{base}} = -j4.0 \times 5.0 = -j20.0 \ \text{A}$$

$$\bar{S}_{\text{sc}} = \bar{S}_{\text{sc.p.u.}} \times S_{\text{base}} = j4.0 \times 1000 = j4000 \ \text{VA}$$

**Transformers and Per Unit System**

A transformer is rated 2000 VA, 200V/400V, and has an internal impedance of j4.0 Ω as seen from the low voltage side.
Figure 4 – Basic Circuit: Example of Per Unit Calculations for a Transformer

(a) Transformer Impedance Referred to the Low Voltage Side
(b) Transformer Impedance Referred to the High Voltage Side

The internal impedance of the transformer as seen from the high voltage side is

\[ \tilde{Z}_{HV} = \tilde{Z}_{LV} \left( \frac{V_2}{V_1} \right)^2 = \text{j}4.0 \times \left( \frac{400}{200} \right)^2 = \text{j}6.0 \ \Omega \]

The rated values for power and voltage are used as the bases for the calculations. It means that the voltage base is different on each side of the transformer. Comparison of the bases and the per unit value on both sides of the transformer is in Table 1.
Table 1 – Bases for transformer low side and high side

Notice in Table 1 that the transformer per unit impedance is the same, regardless of to which side of the transformer it is referred. Again, the conversion complications are absorbed into the base relationships. In the transformer equivalent circuit the different voltage levels disappear and the transformer equivalent circuit is reduced to a single impedance:

Figure 5 – Equivalent Circuit for a Transformer in Per Unit Analysis

Three Phase System and Per Unit Calculations

In three phase systems the total apparent power is given by \( S_{3p} = \sqrt{3} V_{LL} I^*_L \)
where \( V_{LL} \) is the line-to-line voltage and \( I_L \) is the line current.
Similarly, \( S_{3\phi \text{ base}} = \sqrt{3} V_{L \text{ base}} I_{L \text{ base}} \) where

- \( S_{3\phi \text{ base}} \) is the chosen three phase base power, usually the rated power of the equipment, or the chosen base power used for all system calculation by the power utility,
- \( V_{L \text{ base}} \) is the chosen line-to-line base voltage, usually the rated voltage of the equipment, or the nominal voltage level of a part of the power system,
- \( I_{L \text{ base}} \) is the line base current, usually calculated from the base power and the base voltage.

Note that \( S_{3\phi} = 3 S_{\phi} \) and \( S_{3\phi \text{ base}} = 3 S_{\phi \text{ base}} \); thus the p.u. representation remains the same!

**Δ and Y Connection in Per Unit Calculations**

![Figure 6 - Δ - Y Conversion](image)

The base values are related through the same relationships as the actual quantities:

\[
Z_{\Delta \text{ base}} = \frac{V_{L \text{ base}}}{I_{\Delta \text{ base}}} = \sqrt{3} \frac{V_{LN \text{ base}}}{I_{L \text{ base}}} = 3 \frac{V_{LN \text{ base}}}{I_{L \text{ base}}} = 3 Z_{Y \text{ base}}
\]

The per unit values of the \( \Delta \) connected impedances and the \( Y \) connected impedances are
\[ Z_{\Delta \text{ p.u.}} = \frac{Z_{\Delta}}{Z_{\Delta \text{ base}}} \quad Z_{Y \text{ p.u.}} = \frac{Z_{Y}}{Z_{Y \text{ base}}} \]

From that it is easy to show that the per unit value for the \( \Delta \) connection is the same as the per unit value for the \( Y \) connection.

\[ Z_{\Delta \text{ p.u.}} = \frac{Z_{\Delta}}{Z_{\Delta \text{ base}}} = \frac{3Z_{Y}}{3Z_{Y \text{ base}}} = \frac{Z_{Y}}{Z_{Y \text{ base}}} = Z_{Y \text{ p.u.}} \]

**Change of Base**

When pieces of equipment with various different ratings are connected to a system, it is necessary to convert their impedances to a per unit value expressed on the same base. The base that we are converting from will be denoted by subscript \( M \), the base we are converting to will be denoted by subscript \( N \). The base impedance for the bases \( M \) and \( N \) are, respectively,

\[ Z_{M \text{ base}} = \frac{V_{M \text{ base}}^2}{S_{M \text{ base}}} \quad Z_{N \text{ base}} = \frac{V_{N \text{ base}}^2}{S_{N \text{ base}}} \]

The per unit impedances on the bases \( M \) and \( N \) are, respectively

\[ Z_{M \text{ p.u.}} = \frac{Z}{Z_{M \text{ base}}} \quad Z_{N \text{ p.u.}} = \frac{Z}{Z_{N \text{ base}}} \]

where \( Z \) is the actual ohmic value of the impedance of the equipment. It follows that

\[ Z = Z_{M \text{ p.u.}} \cdot Z_{M \text{ base}} = Z_{N \text{ p.u.}} \cdot Z_{N \text{ base}} \]

Substituting for the base impedances we get
\[
Z_{M \text{ p.u.}} \frac{V_{M \text{ base}}^2}{S_{M \text{ base}}} = Z_{N \text{ p.u.}} \frac{V_{N \text{ base}}^2}{S_{N \text{ base}}}
\]

\[
\therefore Z_{N \text{ p.u.}} = Z_{M \text{ p.u.}} \frac{S_{N \text{ base}}}{S_{M \text{ base}}} \frac{V_{M \text{ base}}^2}{V_{N \text{ base}}^2}
\]

Using the MVA and kV notation,

\[
Z_{N \text{ p.u.}} = Z_{M \text{ p.u.}} \frac{\text{MVA}_{N \text{ base}}}{\text{MVA}_{M \text{ base}}} \frac{(\text{kV}_{M \text{ base}})^2}{(\text{kV}_{N \text{ base}})^2}
\]

Accordingly, we can conclude that the per unit system has the following advantages:

- Normalized values against a base have more uniform values in the same system. Thus, it is easier to spot errors. For example, 1000% p.u. will trigger a warning and thus is worth looking into for potential errors.

- Power base is the same through the system; voltage bases are changing according to the transformer turn ratio. Accordingly, ideal transformers’ turn ratios disappear from the per unit system, the conversion complications are absorbed by the base relationships.

- \(\Delta\) base and Y base quantities have the same p.u. values in these two different bases; again the conversion complications are absorbed by the base relationships.

- Three-phase and single-phase quantities have the same p.u. values in these two different bases; again the conversion complications are absorbed by the base relationships.

- Abnormal operating conditions can be easily spotted from the p.u. values.
The bases for different sections in the system can be calculated, carefully verified and stored once for all. When generation/load changes, we only need to change its per unit values and calculate the per unit values accordingly. The conversion will become more reliable. The advantages are more pronounced for large system applications and systems that have many load/generation changes!