One formula sheet allowed. All problems carry equal weight.  
Given: November 23, 1998  

**Problem 1:**  
Consider binary signaling through an additive white Gaussian noise channel. The spectral density of the noise is known from measurements to be equal to \( N_0 / 2 = 10^{-12} \). The received signal is given by  
\[
r(t) = s_i(t) + n(t), \quad 0 \leq t \leq T, \quad i = 1,2
\]
where the two modulation signals are given in the figure below:

![Diagram of modulation signals](image)

The received signal amplitude \( A \) (which is a function of the transmitted power and distance to the receiver among other things) is known to be limited to, \( A \leq 0.001 \), and \( B \) and \( C \) are design parameters. If an error-rate of not more than \( 10^{-6} \) is desired, what is the theoretically maximum bit-rate achievable under the above constraints? In designing for the appropriate error-rate you can use the Chernoff-bound to the error-probability. Show your work.

_______________________________Solution_______________________________

For the above two signals, the best performance (which is needed for maximum bit-rate) is obtained when the signals become orthogonal. This happens when \( B=C=T/2 \). For orthogonal signals, the Chernoff bound implies  
\[
e^{-\frac{E}{2N_0}} \leq 10^{-6} \Rightarrow E \geq 12 \ln(10) \cdot N_0 = 24 \ln(10) \cdot 10^{-12}
\]
We have  
\[
E = A^2 T / 2 \geq 24 \ln(10) \cdot 10^{-12} \Rightarrow 
\]
\[
R = \frac{1}{T} \leq \frac{A^2}{48 \ln(10) \cdot 10^{-12}} \leq \frac{10^{-6}}{48 \ln(10) \cdot 10^{-12}} = 9047.8 \text{ bps}
\]
**Problem 2:**
Consider the following **orthonormal** vectors \( \varphi_1(t), \varphi_2(t) \) and \( \varphi_3(t) \):

![Orthogonal Vectors](image)

(a) Find the signal-space representations of the following two signals:

![Signal Waveforms](image)

(b) For \( T=1 \), plot the waveforms corresponding to the following two signal-space representations: \( x = (1 \ 1 \ 1) \), \( y = (-1 \ 0 \ 1) \).

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**Solution**

(a) In order to make the 3 signals orthonormal, their peak amplitude must equal \( 1/\sqrt{T} \). Then:

\[
\begin{align*}
    s_{11} &= \int_0^T s_1(t) \cdot \varphi_1(t) \, dt = \sqrt{T} \cdot \frac{1}{2} \\
    s_{12} &= \int_0^T s_1(t) \cdot \varphi_2(t) \, dt = \sqrt{T} \cdot \frac{1}{2} \\
    s_{13} &= \int_0^T s_1(t) \cdot \varphi_3(t) \, dt = 0 \\
    \Rightarrow s_1 &= \sqrt{T} \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \\
    s_{21} &= \int_0^T s_2(t) \cdot \varphi_1(t) \, dt = 0 \\
    s_{22} &= \int_0^T s_2(t) \cdot \varphi_2(t) \, dt = \sqrt{T} \cdot \frac{1}{2} \\
    s_{23} &= \int_0^T s_2(t) \cdot \varphi_3(t) \, dt = \sqrt{T} \cdot \frac{1}{2} \\
    \Rightarrow s_2 &= \sqrt{T} \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}
\]

(b)
Problem 3:
Consider the two constellations A and B below, where in constellation A neighboring signals are at a distance $x$ from one-another.

a) For high signal-to-noise ratios, and for the same average energy per bit, compare constellation A to constellation B and determine how many dB one is better than the other.

b) Repeat the comparison in a), but now for the same peak-energy per bit, instead of the same average energy per bit.

Solution

a) 
$$E_A = \frac{5}{4} x^2 \quad \left( d_{\min} \right)^2 = x^2 \Rightarrow \left( d_{\min} \right)^2 = \frac{4}{5} E_A$$

$$E_B = \frac{3}{4} r^2 \quad \left( d_{\min} \right)^2 = r^2 \Rightarrow \left( d_{\min} \right)^2 = \frac{4}{3} E_B$$

Thus, for the same average energy, constellation B is better than constellation A by

$$10\log_{10} \left( \frac{4 E_B}{3 E_A} \right) = 10\log_{10} \left( \frac{5}{3} \right) \approx 2.2 \text{ dB}$$

b) 
$$E_{pA} = \frac{9}{4} x^2 \Rightarrow \left( d_{\min} \right)^2 = \frac{4}{9} E_{pA}$$

$$E_{pB} = r^2 \Rightarrow \left( d_{\min} \right)^2 = E_{pB}$$

Thus, constellation B is better than constellation A for the same peak power by

$$10\log_{10} \left( \frac{1}{4} \right) = 3.52 \text{ dB}$$
Problem 4:
Consider 4-ary signaling over an additive white Gaussian noise (AWGN) channel of spectral density $N_0/2$:

$$r(t) = s_i(t) + n(t), \quad i = 1, 2, 3, 4, \quad 0 \leq t \leq T.$$ 

The four modulation signals are as described in the figure below:

a) Design the simplest possible receiver that processes the received data to make an optimum symbol decision. Explain all important steps.

b) Derive the Union-Chernoff bound to the performance of the optimum receiver above.

Solution

a) All signals have the same energy, so a simple correlation receiver is optimum:

$$\max_{i=1,2,3,4} \int_0^T r(t)s_i(t)dt = \frac{iT}{4} \int (i-1)T/4 r(t)dt$$

The four likelihood statistics can be simply computed by passing $r(t)$ through a filter matched to a rectangular pulse of duration $T/4$ and then sampling its output at $0, T/4, T/2$ and $3T/4$.

b) 

$$P(e) \leq 3e^{\frac{d^2}{4N_0}} = 3e^{\frac{A^2}{8N_0}}$$